Notes on convexity and quanto adjustments for interest rates and related options *

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Abstract

We collect simple and pragmatic exact formulae for the convexity adjustment of irregular interest rate cash flows as Libor-in-arrears or payments of a swap rate (CMS rate) at an irregular date. The results are compared with the results of an approximative approach available in the popular literature.
For options on Libor-in-arrears or CMS rates like caps or binaries we derive an additional new convexity adjustment for the volatility to be used in a standard Black & Scholes and Bachelier model. We study the quality of the adjustments comparing the results of the approximative Black & Scholes formula with the results of an exact valuation formula.
Further we investigate options to exchange interest rates which are possibly set at different dates or admit different tenors.
We collect general quanto adjustments formulae for variable interest rates to be paid in foreign currency and derive valuation formulae for standard options on interest rates paid in foreign currency.

Key words: interest rate options, convexity, quanto adjustment, change of numeraire

JEL Classification: G13

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Contents

1 Introduction

Increasing the return of a variable interest rate investment or cheapening the costs of borrowing can often be achieved by taking advantage of certain shapes of the forward yield curve. A well known and frequently used technique is a delayed setting of the variable index (Libor-in-arrears) or the use of long dates indices like swap rates in place of Libor (CMS rate products) or the use of foreign floating interest rate indices to be paid in domestic currency. It is well known that payments of variable interest rates like Libor or swap rates at dates different from their natural payment dates or in currencies different from their home currency imply certain convexity effects which result in adjusted forward rates.
Pricing those structures in the framework of a fully calibrated term structure model would take the convexity effects automatically into account. However, for many applications this seems to be a modelling overkill. Many convexity adjustment formulae are available in the literature (Hull (2009), Pelsser (2000), Brigo and Mercurio (2006)), some of them based on more or less theoretically sound arguments. Here we collects simple and pragmatic exact formulae for the convexity adjustment for arbitrary irregular interest rate cash flows and compare the results with the outcomes of an approximative approach available in the popular literature.
Then we extend our analysis to options like caps, floors or binaries on irregular interest rates. Given the market model of log-normality for standard interest rate options the consistent model for options on irregular rates is certainly different from a log-normal one. We derive a new additional volatility adjustment that allows an approximation of the true distribution by a log-normal distribution with the same second moment. The results of the approximative pricing are then compared with the exact but more involved valuation. The advantage of the volatility adjustment is that standard pricing libraries can be applied with good accuracy in this context as well. It seems that this new volatility adjustment is so far ignored in practice.
Another application consists of options to exchange interest rates, e.g. Libor-in-arrears versus Libor. For example, those options are implicitly contained in structures involving the maximum or minimum of Libor-in-arrears and Libor.
The final section of this notes collects general formulae for quanto adjustments on floating interest rates paid in a different currency and related options.

2 Notation

Denote by $B(t,T)$ the price of a *zero bond* with maturity T at time $t \le T$. The zero bond pays one unit at time T, $B(T,T) = 1$. At time $t = 0$ the zero bond price $B(0,T)$ is just the discount factor for time T.
The *Libor* $L(S,T)$ for the interval $[S,T]$ is the money market rate for this interval as fixed in the market at time S, this means

$$\begin{aligned} B(S,T) &= \frac{1}{1 + L(S,T)\Delta} \\ L(S,T) &= \frac{\frac{1}{B(S,T)} - 1}{\Delta}, \end{aligned}$$

with Δ as the length of the period $[S,T]$ in the corresponding day count convention.
The *forward Libor* $L^0(S,T)$ for the period $[S,T]$ as seen from today, $t = 0$, is given by

$$L^0(S,T) = \frac{\frac{B(0,S)}{B(0,T)} - 1}{\Delta}. \tag{1}$$

The *swap rate* or *CMS rate* X for a swap with reference dates $T_0 < T_1 < \cdots < T_n$ as fixed at time T_0 is defined as

$$X = \frac{1 - B(T_0, T_n)}{\sum_{i=1}^{n} \Delta_i B(T_0, T_i)}, \tag{2}$$

with Δ_i as length of the period $[T_{i-1}, T_i]$ in the corresponding day count convention. The tenor of X is the time $T_n - T_0$.
The *forward swap rate* X^0 as seen from today is then

$$X^0 = \frac{B(0,T_0) - B(0,T_n)}{\sum_{i=1}^{n} \Delta_i B(0,T_i)}. \tag{3}$$

In the theory of derivative pricing the notion of a numeraire pair $(N, \mathbf{Q}_N)$ plays a central role. In the context of interest rate derivatives the basic securities are the zero bonds of all maturities. A *numeraire pair* (N, Q_N) then consists of a non-negative process N and an associated probability distribution $\mathbf{Q}_N$ such that all basic securities are martingales under $\mathbf{Q}_N$ if expressed in the numeraire N as base unit, i.e.,

$$\frac{B(t,T)}{N_t}, \; t \le T, \text{ is a } \mathbf{Q}_N \text{ martingale for all } T > 0. \tag{4}$$

The *price* $V_0(Y)$ today for a contingent claim[1] Y to be paid at time p is then

$$V_0(Y) = N_0 \mathbf{E}_{\mathbf{Q}_N} \left(\frac{Y}{N_p} \right). \tag{5}$$

For two numeraire pairs $(N, \mathbf{Q}_N)$ and $(M, \mathbf{Q}_M)$ the transformation (Radon-Nikodym density) between the distributions $\mathbf{Q}_N$ and $\mathbf{Q}_M$ is

$$\mathrm{d}\mathbf{Q}_M = \frac{N_0 M_p}{M_0 N_p} \mathrm{d}\mathbf{Q}_N. \tag{6}$$

The time T *forward measure* is the distribution $\mathbf{Q}_T$ referring to the numeraire being the zero bond with maturity T, $N_t = B(t,T)$, and we write

$$\mathbf{Q}_T = \mathbf{Q}_{B(.,T)}. \tag{7}$$

An immediate consequence of (5) applied with $B(.,p)$ as numeraire is

$$V_0(Y) = B(0,p) \mathbf{E}_{\mathbf{Q}_p} Y, \tag{8}$$

i.e., the price of the claim Y today is its discounted expectation under the time p forward measure.

3 Convexity adjusted forward rates

3.1 Libor-in-arrears

In standard interest rate derivatives on Libor the claim on the Libor $L(S,T)$ for period $[S,T]$ is paid at the end of the interval, i.e., at time T. Since the Libor is set at time S this is named "set in advance, pay in arrears". Under $\mathbf{Q}_T$ the process $\frac{B(.,S)}{B(.,T)}$ is a martingale and we get

$$\mathbf{E}_{\mathbf{Q}_T} L(S,T) = \mathbf{E}_{\mathbf{Q}_T} \left(\frac{\frac{B(S,S)}{B(S,T)} - 1}{\Delta} \right) = \frac{\frac{B(0,S)}{B(0,T)} - 1}{\Delta} = L^0(S,T). \tag{9}$$

This implies from (8) the well-known expression for the price of a Libor,

$$V_0(L(S,T)) = B(0,T) L^0(S,T), \tag{10}$$

as discounted forward Libor.

[1] We assume that Y can be hedged with the basic securities.

In a Libor-in-arrears payment the Libor $L(S,T)$ for period $[S,T]$ is now paid at the time S of its fixing. According to (8) its price today is

$$B(0,S)\mathbf{E}_{\mathbf{Q}_s}(L(S,T)). \tag{11}$$

Our goal is to express $\mathbf{E}_{\mathbf{Q}_s}(L(S,T))$ in terms of the forward rate $L^0(S,T)$ plus some "convexity" adjustment, the convexity charge.

Proposition 1 *The following general valuation formula holds*

$$\boxed{\mathbf{E}_{\mathbf{Q}_s}(L(S,T)) = L^0(S,T)\left(1+\frac{\Delta}{L^0(S,T)}\frac{\mathrm{Var}_{\mathbf{Q}_T}L(S,T)}{1+\Delta L^0(S,T)}\right),} \tag{12}$$

with $\mathrm{Var}_{\mathbf{Q}_T}L(S,T)$ *as the variance of* $L(S,T)$ *under the distribution* $\mathbf{Q}_T$.

Proof: Using (6) we obtain

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}_s}(L(S,T)) &= \mathbf{E}_{\mathbf{Q}_T}\left(L(S,T)\frac{B(S,S)B(0,T)}{B(S,T)B(0,S)}\right) \\
&= \mathbf{E}_{\mathbf{Q}_T}\left(L(S,T)(1+\Delta L(S,T))\frac{B(0,T)}{B(0,S)}\right) \\
&= \frac{\mathbf{E}_{\mathbf{Q}_T}((L(S,T)+\Delta L(S,T)^2)}{1+\Delta L^0(S,T)} \\
&= \frac{(L^0(S,T)+\Delta \mathrm{Var}_{\mathbf{Q}_T}L(S,T)+\Delta L^0(S,T)^2)}{1+\Delta L^0(S,T)} \\
&= L^0(S,T)\left(1+\frac{\Delta}{L^0(S,T)}\frac{\mathrm{Var}_{\mathbf{Q}_T}L(S,T)}{1+\Delta L^0(S,T)}\right).
\end{aligned}$$

$\diamondsuit$

Under the so-called market model which is the model underlying the market valuation for caps, the Libor $L(S,T)$ is lognormal under $\mathbf{Q}_T$ with volatility σ,

$$L(S,T) = L^0(S,T)\exp(\sigma W_S - \frac{1}{2}\sigma^2 S), \tag{13}$$

with some Wiener process (W_t). In this case the convexity adjustment can be expressed in terms of the volatility and (12) reduces to

$$\boxed{\mathbf{E}_{\mathbf{Q}_s}(L(S,T)) = L^0(S,T)\left(1+\frac{\Delta L^0(S,T)(\exp(\sigma^2 S)-1)}{1+\Delta L^0(S,T)}\right).} \tag{14}$$

Under the Bachelier/normal model,

$$L(S,T) = L^0(S,T) + \sigma_n W_S, \tag{15}$$

with normal volatility σ_n, the valuation formula is

$$\mathbf{E}_{\mathbf{Q}_s}\left(L(S,T)\right) = L^0(S,T)\left(1 + \frac{\Delta}{L^0(S,T)}\frac{\sigma_n^2 S}{(1+\Delta L^0(S,T))}\right). \tag{16}$$

Example. To illustrate the magnitude of the convexity charge we show below as an example the convexity adjusted forward rates $\mathbf{E}_{\mathbf{Q}_s}\left(L(S,T)\right)$ for various maturities S and volatilities σ. We assume $L^0(S,T) = 5\%$ and $\Delta = 0.5$.

S/σ	10%	15%	20%
1	5,001%	5,003%	5,005%
2	5,002%	5,006%	5,010%
3	5,004%	5,009%	5,016%
4	5,005%	5,011%	5,021%
5	5,006%	5,015%	5,027%
6	5,008%	5,018%	5,033%
7	5,009%	5,021%	5,039%
8	5,010%	5,024%	5,046%
9	5,011%	5,027%	5,053%
10	5,013%	5,031%	5,060%

In the general case, for arbitrary payment times $p \geq S$ a somewhat more involved formula can be derived, see Schmidt (1996). However, using an idea similar to the assumption of a linear swap rate model below, we can easily derive a formula even for $p \geq S$ following the same line of arguments as above.
Let us assume a linear model of the from

$$\frac{B(S,p)}{B(S,T)} = \alpha + \beta_p L(S,T),\ \forall p \geq S. \tag{17}$$

The constants α and β_p are straightforward to determine in order to make the model consistent. Since $\frac{B(.,p)}{B(.,T)}$ is a $\mathbf{Q}_T$ martingale and using (9)

$$\begin{aligned}\frac{B(0,p)}{B(0,T)} &= \mathbf{E}_{\mathbf{Q}_T}\frac{B(S,p)}{B(S,T)}\\ &= \mathbf{E}_{\mathbf{Q}_T}(\alpha + \beta_p L(S,T))\\ &= (\alpha + \beta_p L^0(S,T)),\end{aligned}$$

which implies $\beta_p = \left(\frac{B(0,p)}{B(0,T)} - \alpha\right)/L^0(S,T)$. Also $\alpha = 1$ as a consequence of $1 = \frac{B(S,T)}{B(S,T)} = \alpha + \beta_T L(S,T)$, so finally,

$$\beta_p = \left(\frac{B(0,p)}{B(0,T)} - 1\right)/L^0(S,T). \tag{18}$$

Now we can formulate the result for $p \geq S$.

Proposition 2 *Under the assumption of a linear Libor model (17) we have the following general formula for payments of the Libor* $L(S,T)$ *at arbitrary times* $p \geq S$

$$\boxed{\mathbf{E}_{\mathbf{Q}_p}\left(L(S,T)\right) = L^0(S,T)\left(1 + \frac{1 - \frac{B(0,T)}{B(0,p)}}{(L^0(S,T))^2}\mathrm{Var}_{\mathbf{Q}_T} L(S,T)\right),} \tag{19}$$

with $\mathrm{Var}_{\mathbf{Q}_T} L(S,T)$ *as the variance of* $L(S,T)$ *under the distribution* $\mathbf{Q}_T$.

Remarks.
1. For $p = S$ formula (19) obviously reduces to the general formula (12).
2. In case of the market model (13) for caps formula (19) gets more explicit:

$$\boxed{\mathbf{E}_{\mathbf{Q}_p}\left(L(S,T)\right) = L^0(S,T)\left(1 + \left(1 - \frac{B(0,T)}{B(0,p)}\right)(\exp(\sigma^2 S) - 1)\right)} \tag{20}$$

3. For the normal (Bachelier) model (15),

$$\mathbf{E}_{\mathbf{Q}_p}\left(L(S,T)\right) = L^0(S,T)\left(1 + \frac{1 - \frac{B(0,T)}{B(0,p)}}{(L^0(S,T))^2}\sigma_n^2 S\right). \tag{21}$$

3.2 CMS

The market standard valuation formula for swaptions is closely related to a particular numeraire pair called the *swap numeraire* or *PV01 numeraire* pair $(\mathrm{PV01}, \mathbf{Q}_{\mathrm{Swap}})$ with numeraire

$$N_t = \mathrm{PV01}_t = \sum_{i=1}^{n} \Delta_i B(t, T_i),\ t \leq T_1. \tag{22}$$

Under $\mathbf{Q}_{\mathrm{Swap}}$ the expectation of the swap rate X is just the forward swap rate X^0 which is again a consequence of the martingale property of $B(., T_0), B(., T_n)$ if expressed in the numeraire PV01

$$\begin{aligned}
\mathbf{Q}_{\mathrm{Swap}} X &= \mathbf{Q}_{\mathrm{Swap}}\left(\frac{B(T_0,T_0) - B(T_0,T_n)}{\sum_{i=1}^{n}\Delta_i B(T_0,T_i)}\right) \\
&= \mathbf{Q}_{\mathrm{Swap}}\left(\frac{B(T_0,T_0) - B(T_0,T_n)}{\mathrm{PV01}_{T_0}}\right) \\
&= \frac{B(0,T_0) - B(0,T_n)}{\mathrm{PV01}_0} = X^0.
\end{aligned} \tag{23}$$

In a CMS based security, e.g., a CMS swap or cap, the rate X is paid only once and at a time $p \geq T_0$. According to (8) we are interested in an explicit valuation of

$$\mathbf{E}_{\mathbf{Q}_p}(X)$$

in terms of the forward swap rate X^0 and some "convexity" adjustment.
Applying (6) we get

$$\mathbf{E}_{\mathbf{Q}_p}(X) = \mathbf{E}_{\mathbf{Q}_{\text{Swap}}}\left(X \frac{B(T_0,p)}{\text{PV01}_{T_0}} \frac{\text{PV01}_0}{B(0,p)}\right) = \frac{\text{PV01}_0}{B(0,p)} \mathbf{E}_{\mathbf{Q}_{\text{Swap}}}\left(X \frac{B(T_0,p)}{\text{PV01}_{T_0}}\right). \tag{24}$$

In order to calculate the right hand side explicitly one has to express or approximate $\frac{B(T_0,p)}{\text{PV01}_{T_0}}$ in terms of simpler objects like Libor or the swap rate X itself. Several approximations are studied in Schmidt (1996). We rely here on a very elegant approximation based the assumption of a *linear swap rate model*, see Hunt and Kennedy (2004) or Pelsser (2000),

$$\frac{B(T_0,p)}{\text{PV01}_{T_0}} = \alpha + \beta_p \cdot X,\ p \geq T_0. \tag{25}$$

The constant α and the factor β_p have to be determined consistently. Since $\frac{B(.,p)}{\text{PV01}_.}$ is a $\mathbf{Q}_{\text{Swap}}$-martingale using (23) we obtain

$$\begin{aligned}
\frac{B(0,p)}{\text{PV01}_0} &= \mathbf{E}_{\mathbf{Q}_{\text{Swap}}}\left(\frac{B(T_0,p)}{\text{PV01}_{T_0}}\right) \\
&= \mathbf{E}_{\mathbf{Q}_{\text{Swap}}}(\alpha + \beta_p X) \\
&= (\alpha + \beta_p X^0),
\end{aligned}$$

and thus

$$\beta_p = \frac{\frac{B(0,p)}{\text{PV01}_0} - \alpha}{X^0}. \tag{26}$$

To determine α observe that

$$\begin{aligned}
1 &= \frac{\sum_{i=1}^n \Delta_i B(T_0,T_i)}{\text{PV01}_{T_0}} \\
&= \sum_{i=1}^n \Delta_i \alpha + \sum_{i=1}^n \Delta_i \beta_{T_i} X \\
&= \sum_{i=1}^n \Delta_i \alpha + (1 - \sum_{i=1}^n \Delta_i \alpha) \frac{X}{X^0},
\end{aligned}$$

which yields

$$\alpha = \frac{1}{\sum_{i=1}^n \Delta_i}. \tag{27}$$

Proposition 3 *Under the assumption (25) of the linear swap rate model we have the following general valuation formula*

$$\boxed{\mathbf{E}_{\mathbf{Q}_p} X = X^0 \left(1 + \frac{1 - \frac{(B(0,T_0) - B(0,T_n))}{X^0 B(0,p) \sum_{i=1}^{n} \Delta_i}}{(X^0)^2} \mathrm{Var}_{\mathbf{Q}_{\mathrm{Swap}}}(X) \right),} \tag{28}$$

with $\mathrm{Var}_{\mathbf{Q}_{\mathrm{Swap}}}(X)$ *as the variance of* X *under the swap measure* $\mathbf{Q}_{\mathrm{Swap}}$.

Proof: Using (24), (25) and (23) we obtain

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}_p} X &= \frac{\mathrm{PV01}_0}{B(0,p)} \mathbf{E}_{\mathbf{Q}_{\mathrm{Swap}}} \left(X(\alpha + \beta_p X) \right) \\ &= \frac{1}{\alpha + \beta_p X^0} (\alpha X^0 + \beta_p (X^0)^2 + \beta_p \mathrm{Var}_{\mathbf{Q}_{\mathrm{Swap}}}(X)) \\ &= X_0 \left(1 + \frac{\beta_p \mathrm{Var}_{\mathbf{Q}_{\mathrm{Swap}}}(X))}{X^0(\alpha + \beta_p X^0)} \right). \end{aligned}$$

Substituting

$$\alpha + \beta_p X_0 = \frac{B(0,p)}{\mathrm{PV01}_0} = \frac{B(0,p) X^0}{(B(0,T_0) - B(0,p))}$$

and using (27) yields the assertion. ◇

Remark. In the special case of a one period swap the CMS adjustment formula (28) reduces to the respective formula (19) for Libor payments.

Under the market model for swaptions it is known that the swap rate X is lognormal under $\mathbf{Q}_{\mathrm{Swap}}$,

$$X = X^0 \exp(\sigma W_{T_0} - \frac{1}{2}\sigma^2 T_0), \tag{29}$$

with some Wiener process (W_t). In this case the variance of X under $\mathbf{Q}_{\mathrm{Swap}}$ is just

$$\mathrm{Var}_{\mathbf{Q}_{\mathrm{Swap}}}(X) = (X^0)^2 (\exp(\sigma^2 T_0) - 1) \tag{30}$$

and (28) reduces to

$$\boxed{\mathbf{E}_{\mathbf{Q}_p} X = X^0 \left(1 + \left(1 - \frac{B(0,T_0) - B(0,T_n)}{X^0 B(0,p) \sum_{i=1}^{n} \Delta_i} \right) (\exp(\sigma^2 T_0) - 1) \right).} \tag{31}$$

For the normal (Bachelier) model,

$$X = X^0 + \sigma_n W_{T_0}, \tag{32}$$

with normal volatility σ_n, the variance in (28) is obviously

$$\mathrm{Var}_{\mathbf{Q}_{\mathrm{Swap}}}(X) = \sigma_n^2 T_0.$$

Example. Again we illustrate the size of the CMS adjustment by an example. We assume that the swap curve is flat at 5% for all maturities. Payment of the CMS rate is at $p = T_0 + 1$ as it happens in most cases in practice.

σ	10,00%			15%			20%		
$T_0/T_n - T_0$	5	10	20	5	10	20	5	10	20
1	5,005%	5,010%	5,017%	5,010%	5,022%	5,039%	5,019%	5,039%	5,071%
2	5,009%	5,019%	5,035%	5,021%	5,044%	5,080%	5,038%	5,079%	5,144%
3	5,014%	5,029%	5,053%	5,032%	5,066%	5,121%	5,058%	5,121%	5,220%
4	5,019%	5,039%	5,071%	5,043%	5,089%	5,163%	5,079%	5,164%	5,300%
5	5,023%	5,048%	5,089%	5,054%	5,113%	5,206%	5,101%	5,209%	5,382%
6	5,028%	5,058%	5,107%	5,066%	5,137%	5,250%	5,123%	5,256%	5,469%
7	5,033%	5,068%	5,125%	5,077%	5,161%	5,295%	5,147%	5,305%	5,558%
8	5,038%	5,079%	5,144%	5,089%	5,186%	5,341%	5,170%	5,356%	5,652%
9	5,043%	5,089%	5,163%	5,101%	5,212%	5,388%	5,196%	5,409%	5,749%
10	5,048%	5,099%	5,182%	5,114%	5,238%	5,436%	5,222%	5,464%	5,850%

Proposition 3 yields an interesting corollary.

Corollary 1 *For payment times p ranging from T_0 to T_n the convexity charge*

$$C(p) = X^0 \left(\frac{1 - \frac{(B(0,T_0) - B(0,T_n))}{X^0 B(0,p) \sum_{i=1}^n \Delta_i}}{(X^0)^2} \mathrm{Var}_{\mathbf{Q}_{\mathrm{Swap}}}(X) \right)$$

is monotonously decreasing in p changing its sign from positive at $p = T_0$ to negative at $p = T_n$. The charge changes sign and vanishes exactly at the point $p \in (T_0, T_n)$ where $B(0,p) \sum_{i=1}^n \Delta_i = \sum_{i=1}^n \Delta_i B(0, T_i)$. Moreover,

$$\sum_{i=1}^n C(T_i) = 0.$$

Intuitively, this is not surprising, since applying the valuation formula (28) multiplied with $B(0, p)$ for all $p = T_1, \ldots, T_n$ and summing up we end up with the valuation of a full interest rate swap and all convexity adjustments should cancel out.

Observe that the qualitative statement of the Corollary remains true also without the assumption of a linear swap rate model, see Schmidt (1996).

3.3 Unified approach under the linear rate model

Assuming a linear rate model (17) or (25) we derived a closed valuation formula for a Libor or CMS rate paid at an arbitrary date. Both derivations can be unified under one umbrella.
Write Y_S for a floating rate which is set at time S. Examples of particular interest are $Y_S = L(S,T)$, the Libor for the interval $[S,T]$, or, $Y_S = X$ with $S = T_0$ and X the swap rate with reference dates $T_0 < T_1 < \cdots < T_n$. Let $N, \mathbf{Q}_N$ denote the natural ("market") numeraire pair associated with Y_S and all we need is that

$$\mathbf{E}_{\mathbf{Q}_N} Y_S = Y_0, \tag{33}$$

where Y_0 is known and a function of the yield curve $B(0,.)$ today.
We are interested in todays price of the rate Y_S to be paid at some time $p \geq S$,

$$B(0,p)\mathbf{E}_{\mathbf{Q}_p} Y_S.$$

Assume a linear rate model of the form

$$\frac{B(S,p)}{N_S} = \alpha + \beta_p Y_S, \tag{34}$$

with some deterministic α, β_p which have to be determined accordingly to make the model consistent, see (18) resp. (27), (26) for α and β_p in case of a linear Libor resp. swap rate model. From the martingale property of $\frac{B(.,p)}{N.}$ and (33) we get immediately

$$\frac{B(0,p)}{N_0} = \alpha + \beta_p Y_0.$$

Proposition 4 *Under the assumption of a linear model (34) we have the following general valuation formula*

$$\boxed{\mathbf{E}_{\mathbf{Q}_p} Y_s = Y_0 \left(1 + \frac{\beta_p}{Y_0(\alpha + \beta_p Y_0)} \mathrm{Var}_{\mathbf{Q}_N}(Y_S)\right),} \tag{35}$$

with $\mathrm{Var}_{\mathbf{Q}_N}(Y_S)$ *as the variance of* Y_S *under the measure* $\mathbf{Q}_N$.

Proof: On the information up to time S the density of the time p forward measure $\mathbf{Q}_p$ w.r.t. $\mathbf{Q}_N$ is according to (6)

$$d\mathbf{Q}_p = \frac{N_0}{B(0,p)} \frac{B(S,p)}{N_S} d\mathbf{Q}_N = \frac{\alpha + \beta_p Y_S}{\alpha + \beta_p Y_0} d\mathbf{Q}_N. \tag{36}$$

Therefore,

$$
\begin{aligned}
\mathbf{E}_{\mathbf{Q}_p} Y_S &= \mathbf{E}_{\mathbf{Q}_N}\left(Y_S \frac{\alpha + \beta_p Y_S}{\alpha + \beta_p Y_0} \right) \\
&= Y_0 \left(1 + \frac{\beta_p}{Y_0(\alpha + \beta_p Y_0)} \mathrm{Var}_{\mathbf{Q}_N}(Y_S) \right).
\end{aligned}
$$

$\diamondsuit$

In case of a linear Libor model or a linear swap rate model formula (35) reduces to (19) or (28), respectively.

If the distribution of Y_S under $\mathbf{Q}_N$ is lognormal with volatility σ_Y,

$$
Y_S = Y_0 \exp(\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S),
$$

the variance in (35) is

$$
\mathrm{Var}_{\mathbf{Q}_N}(Y_S) = Y_0^2(\exp(\sigma_Y^2 S) - 1). \tag{37}
$$

For the normal (Bachelier) model,

$$
Y_S = Y_0 + \sigma_{Y,n} W_S,
$$

we have

$$
\mathrm{Var}_{\mathbf{Q}_N}(Y_S) = \sigma_{Y,n}^2 S. \tag{38}
$$

3.4 Comparison study

In this section we compare the results of the above convexity adjustment formulae with the results of a popular formula which can be found, for example, in Hull (2009), Section 16.11.
Suppose we are interested in a derivative involving a yield Y at time p of a bond with yield to price function $P(Y)$. To the forward price P_0 as of today for a forward contract maturing at time p there corresponds a forward yield Y_0,

$$
P_0 = P(Y_0).
$$

Then Hull (2009) gives the following approximative formula

$$
\boxed{\mathbf{E}_{\mathbf{Q}_p}(Y) \approx Y_0 - \frac{1}{2} Y_0^2 \sigma^2 p \frac{P''(Y_0)}{P'(Y_0)},} \tag{39}
$$

with σ as the volatility of the yield Y.

Applying this to a Libor-in-arrears, $Y = L(S,T), p = S, P(y) = \frac{1}{1+\Delta y}$ yields the formula

$$\boxed{\mathbf{E}_{\mathbf{Q}_s}\left(L(S,T)\right) \approx L^0(S,T)\left(1 + \frac{\Delta L^0(S,T)\sigma^2 S}{1+\Delta L^0(S,T)}\right),} \tag{40}$$

which is in line with the exact formula (14) as long as one approximates $(\exp(\sigma^2 S) - 1)$ to the first order

$$\exp(\sigma^2 S) - 1) \approx \sigma^2 S.$$

It is obvious that (40) underestimates the true convexity as quantified by (14) which becomes apparent in particular for long dated Libor-in-arrears structures and relatively high volatilities.
For an interest rate level of $L^0(S,T) = 5\%$[2] the following table shows a comparison of the convexity charges as resulting from equation (14) or (40), respectively,

	$\sigma = 10\%$		$\sigma = 20\%$	
S	(14)	(40)	(14)	(40)
1	0,001%	0,001%	0,005%	0,005%
2	0,002%	0,002%	0,010%	0,010%
3	0,004%	0,004%	0,016%	0,015%
4	0,005%	0,005%	0,021%	0,020%
5	0,006%	0,006%	0,027%	0,025%
6	0,008%	0,007%	0,033%	0,030%
7	0,009%	0,009%	0,040%	0,035%
8	0,010%	0,010%	0,046%	0,039%
9	0,012%	0,011%	0,053%	0,044%
10	0,013%	0,012%	0,061%	0,049%
11	0,014%	0,014%	0,068%	0,054%
12	0,016%	0,015%	0,076%	0,059%
13	0,017%	0,016%	0,084%	0,064%
14	0,019%	0,017%	0,093%	0,069%
15	0,020%	0,018%	0,101%	0,074%
16	0,021%	0,020%	0,111%	0,079%
17	0,023%	0,021%	0,120%	0,084%
18	0,024%	0,022%	0,130%	0,089%
19	0,026%	0,023%	0,140%	0,094%
20	0,027%	0,025%	0,151%	0,099%

Now we apply the general formula (39) to the situation of CMS rate. In this case

[2] On an act/360 basis.

$Y = X$ and Y can be interpreted as the yield of a coupon bond with coupon dates $T_1, \dots, T_n$ and coupon $C = X^0$. At time $p = T_0$ the forward bond price P_0 is at par and the corresponding forward yield is $Y_0 = X^0$. For a bond with annual coupons, $T_i = T_0 + i$, and face value of 1 we have

$$
\begin{aligned}
P(Y) &= \sum_{i=1}^{n} \frac{C}{(1+Y)^i} + \frac{1}{(1+Y)^n} \\
P'(Y) &= \sum_{i=1}^{n} \frac{-i\,C}{(1+Y)^{i+1}} - \frac{n}{(1+Y)^{n+1}} \\
P''(Y) &= \sum_{i=1}^{n} \frac{i(i+1)C}{(1+Y)^{i+2}} + \frac{n(n+1)}{(1+Y)^{n+2}}.
\end{aligned}
$$

Observe that the formula (39) can be applied correctly only in case of a payment of the CMS rate at fixing, i.e., $p = T_0$. This is not what is of interest in most practical applications, where usually $p = T_1$. However, ignoring this inconsistency, equation (39) is often applied for $p = T_1$ which yields a higher convexity charge contrary to what should be the case according to the result of Corollary 1. As we shall see below in the numerical examples, equation (39) underestimates the convexity, so overall, luckily the inconsistency corrects some other error.

Comparing the convexity charges of equation (39) applied to the case of a CMS rate on one hand and of equations (28) and (30) on the other hand we get the following results for a flat interest rate environment of $X^0 = 5\%$ and a tenor of 5 years for the underling CMS rate X:

	$\sigma = 10\%$		$\sigma = 20\%$	
$p = T_0$	(39)	(28) & (30)	(39)	(28) & (30)
1	0,138%	0,137%	0,553%	0,558%
2	0,276%	0,271%	1,106%	1,116%
3	0,415%	0,408%	1,659%	1,709%
4	0,553%	0,549%	2,211%	2,332%
5	0,691%	0,687%	2,764%	2,967%
6	0,829%	0,829%	3,317%	3,635%
7	0,968%	0,972%	3,870%	4,330%
8	1,106%	1,116%	4,423%	5,053%
9	1,244%	1,258%	4,976%	5,789%
10	1,382%	1,405%	5,529%	6,570%
11	1,520%	1,553%	6,081%	7,383%
12	1,659%	1,703%	6,634%	8,230%
13	1,797%	1,854%	7,187%	9,111%
14	1,935%	2,007%	7,740%	10,028%
15	2,073%	2,162%	8,293%	10,982%
16	2,211%	2,318%	8,846%	11,975%
17	2,350%	2,475%	9,399%	13,009%
18	2,488%	2,634%	9,951%	14,085%
19	2,626%	2,795%	10,504%	15,205%
20	2,764%	2,958%	11,057%	16,371%

The same analysis repeated for a tenor of 10 years for the CMS rate yields

	$\sigma = 10\%$		$\sigma = 20\%$	
$p = T_0$	(39)	(28) & (30)	(39)	(28) & (30)
1	0,243%	0,230%	0,971%	0,934%
2	0,486%	0,460%	1,943%	1,897%
3	0,728%	0,694%	2,914%	2,904%
4	0,971%	0,929%	3,885%	3,952%
5	1,214%	1,167%	4,856%	5,039%
6	1,457%	1,407%	5,828%	6,173%
7	1,700%	1,650%	6,799%	7,354%
8	1,943%	1,895%	7,770%	8,583%
9	2,185%	2,142%	8,741%	9,855%
10	2,428%	2,392%	9,713%	11,185%
11	2,671%	2,644%	10,684%	12,569%
12	2,914%	2,899%	11,655%	14,011%
13	3,157%	3,157%	12,626%	15,510%
14	3,399%	3,417%	13,598%	17,072%
15	3,642%	3,680%	14,569%	18,696%
16	3,885%	3,946%	15,540%	20,387%
17	4,128%	4,214%	16,511%	22,148%
18	4,371%	4,485%	17,483%	23,980%
19	4,613%	4,759%	18,454%	25,886%
20	4,856%	5,035%	19,425%	27,871%

Again, (39) consistently underestimates the convexity charge, which is particularly significant for very long dated payments and high volatilities.

Overall, for both, the convexity charge in the Libor-in-arrears case and in the CMS case, the size of the charge increases with the time to the payment and with the volatility of the underlying rate.

4 Options on Libor-in-arrears and CMS rates

In this section we investigate European options on interest rates like Libor $L(S,T)$ for period $[S,T]$ or CMS rates X with reference dates $T_0 < T_1 < \cdots < T_n$. The payment date of the option is an arbitrary time point p with $p \geq S$ or $p \geq T_0$, respectively. Of particular interest are caps and floors or binaries. For standard caps and floors on Libor we have $p = T$ and the standard market model postulates a lognormal distribution of $L(S,T)$ under the forward measure $\mathbf{Q}_T$. For standard options on a swap rate X, i.e. swaptions, the market used a lognormal distribution for X under the swap measure $\mathbf{Q}_{\text{Swap}}$. However in the general case, i.e., for

options on Libor or CMS with arbitrary payment date p a lognormal model would be inconsistent with the market model for standard options.
In Sections 4.1 and 4.2 we follow again the general setup of Section 3.3. Y_S is a floating interest rate which is set at time S and $(N, \mathbf{Q}_N)$ denotes the "market" numeraire pair associated with Y_S. We assume that the distribution of Y_S under $\mathbf{Q}_N$ is either lognormal with volatility σ_Y,

$$Y_S = Y_0 \exp(\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S), \tag{41}$$

or normal with volatility $\sigma_{Y,n}$,

$$Y_S = Y_0 + \sigma_{Y,n} W_S. \tag{42}$$

For the payment date $p \geq S$ we assume again the linear rate model

$$\frac{B(S,p)}{N_S} = \alpha + \beta_p Y_S. \tag{43}$$

Recall that for the case of $Y_S = L(S,T), N_S = B(S,T)$ and $p = S$, i.e., the case of Libor-in-arrears, the assumption of a linear model is trivially satisfied and no restriction.

4.1 Volatility adjustments

In this section we derive a simple lognormal approximation for the distribution of a rate to be paid at an arbitrary date p which is based on a suitably adjusted volatility.
The main motivation for this approximation is the desire to be able to use standard valuation formulae also for options on interest rates which are irregularly paid. As we shall see below in Section 4.2 there exists also exact valuation formulae but these are somewhat more involved.

Proposition 5 *Suppose (41) and (43). Then for arbitrary $p \geq S$ under $\mathbf{Q}_p$ the rate Y_S is approximately lognormal*

$$Y_S \approx E_{\mathbf{Q}_p}(Y_s)\ \exp(\sigma_Y^* W_S^* - \frac{1}{2}(\sigma_Y^*)^2 S) \tag{44}$$

with volatility

$$\boxed{(\sigma_Y^*)^2 = \sigma_Y^2 + \ln\left[\frac{(\alpha+\beta_p Y_0)(\alpha+\beta_p Y_0 \exp(2\sigma_Y^2 S))}{(\alpha+\beta_p Y_0\exp(\sigma_Y^2 S))^2}\right]/S,} \tag{45}$$

and $E_{\mathbf{Q}_p}(Y_S)$ given by (35),(37).

Proof: We calculate the second moment of Y_S under $\mathbf{Q}_p$. Using (36) and (41) we obtain

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}_p} Y_S^2 &= \mathbf{E}_{\mathbf{Q}_N} \left(Y_S^2 \frac{\alpha + \beta_p Y_S}{\alpha + \beta_p Y_0} \right) \\
&= \frac{1}{\alpha + \beta_p Y_0} \mathbf{E}_{\mathbf{Q}_N} \left(\alpha Y_S^2 + \beta_p Y_S^3 \right) \\
&= \frac{1}{\alpha + \beta_p Y_0} Y_0^2 \left(\alpha \exp(\sigma_Y^2 S) + \beta_p Y_0 \exp(3\sigma_Y^2 S) \right) .
\end{aligned}$$

In view of (37) we get for the variance of Y_S under $\mathbf{Q}_p$

$$\mathrm{Var}_{\mathbf{Q}_p} Y_S = (\mathbf{E}_{\mathbf{Q}_p} Y_S)^2 \left(\frac{\exp(\sigma_Y^2 S)(\alpha + \beta_p Y_0)(\alpha + \beta_p Y_0 \exp(2\sigma_Y^2 S))}{(\alpha + \beta_p Y_0 \exp(\sigma_Y^2 S))^2} - 1 \right) .$$

Assuming a hypothetical lognormal distribution for Y_S under $\mathbf{Q}_p$ with volatility σ_Y^* as on the right hand side of (44) we would have

$$\mathrm{Var}_{\mathbf{Q}_p} Y_S = (\mathbf{E}_{\mathbf{Q}_p} Y_S)^2 \left(\exp((\sigma_Y^*)^2 S) - 1 \right) .$$

Now matching moments yields the assertion. $\diamondsuit$

Analogously we obtain the corresponding result for the Bachelier model.

Proposition 6 *Suppose (42) and (43). Then for arbitrary $p \geq S$ under $\mathbf{Q}_p$ the rate Y_S is approximately normal*

$$Y_S \approx E_{\mathbf{Q}_p}(Y_s) + \sigma_Y^* W_S^* \tag{46}$$

with volatility

$$\boxed{(\sigma_Y^*)^2 = \sigma_Y^2 \left(1 + \frac{2\beta_p Y_0}{\alpha + \beta_p Y_0} \right)} \tag{47}$$

and $E_{\mathbf{Q}_p}(Y_S)$ given by (35),(38).

4.2 Exact valuation under the linear rate model

Assuming a linear rate model it is relatively straightforward to derive exact valuation formulae for standard European options like calls, puts or binaries on Libor or CMS rates to be paid at an arbitrary date. However, the resulting formulae are somewhat more involved. We are interested in the valuation of standard options on the rate Y_S but the option payout is at some arbitrary time $p \geq S$. The value of a call option with strike K is then

$$\begin{aligned}
&B(0,p)\mathbf{E}_{\mathbf{Q}_p} \max(Y_S - K, 0) \\
&= N_0 \mathbf{E}_{\mathbf{Q}_N} \left(\max(Y_S - K, 0) \frac{B(S,p)}{N_S} \right) \\
&= N_0 \mathbf{E}_{\mathbf{Q}_N} \left(\max(Y_S - K, 0)(\alpha + \beta_p Y_S) \right) .
\end{aligned}$$

This expectation is straightforward to calculate explicitly for the modeling assumptions (41) or (42). In the following we use $\varepsilon = +1$ for a Cap and $\varepsilon = -1$ for a Floor,

Proposition 7 *Under the assumptions (41) and (43) the value of a Cap and Floor option on the rate Y_S with payment at time $p \geq S$ is given by*

$$B(0,p)\mathbf{E}_{\mathbf{Q}_p} \max(\varepsilon(Y_S - K), 0) \tag{48}$$
$$= \frac{B(0,p)}{\alpha + \beta_p Y_0} \varepsilon \left[Y_0 \mathrm{N}(\varepsilon d_1)(\alpha - \beta_p K) - \alpha K \mathrm{N}(\varepsilon d_2) + \beta_p Y_0^2 \mathbf{e}^{\sigma_Y^2 S} \mathrm{N}(\varepsilon(d_1 + \sigma_Y \sqrt{S})) \right],$$

with

$$d_1 = \frac{\ln(\frac{Y_0}{K}) + \frac{1}{2}\sigma_Y^2 S}{\sigma_Y \sqrt{S}}$$
$$d_2 = \frac{\ln(\frac{Y_0}{K}) - \frac{1}{2}\sigma_Y^2 S}{\sigma_Y \sqrt{S}}$$

For the special case of an in-arrears option on a Libor $Y_S = L(S,T)$, $p = S$, the assumption of a linear model is obviously satisfied and the corresponding valuation has been noted in Brigo and Mercurio (2006), Section 13.2.1.

Proposition 8 *Under the assumptions (42) and (43) the value of a Cap and Floor option on the rate Y_S with payment at time $p \geq S$ is given by*

$$B(0,p)\mathbf{E}_{\mathbf{Q}_p} \max(\varepsilon(Y_S - K), 0) \tag{49}$$
$$= \frac{B(0,p)}{\alpha + \beta Y_0} \left[\varepsilon \left\{ (Y_0 - K)(\alpha + \beta Y_0) + \beta \sigma_{Y,n}^2 S \right\} \mathrm{N}(\varepsilon d) + \sigma_{Y,n} \sqrt{S} (\alpha + \beta Y_0) \mathrm{n}(\varepsilon d) \right],$$

with

$$d = \frac{Y_0 - K}{\sigma_{Y,n} \sqrt{S}},$$

and n *denotes the density of the standard normal distribution.*

Remark 1 *When implementing equation (48) and (49) we can distinguish different volatilities:*

1. *The index volatility σ^I which is responsible for the index convexity (cf. (35)). Here typically an ATM volatility is used.*

2. *The strike volatility σ^K used to value the option which is usually taken from some smile information.*

Separating the volatilities has the advantage of creating consistent index convexities for options with different strikes. It is achieved by giving the two terms with Y_S in

$$N_0 \mathbf{E}_{\mathbf{Q}_N} \left(\max(Y_S - K, 0)(\alpha + \beta_p Y_S) \right)$$

a different volatility when calculating the expectation. Applied to formula (48) this yields

$$\begin{aligned} & B(0,p) \mathbf{E}_{\mathbf{Q}_p} \max(Y_S - K, 0) \\ = & \frac{B(0,p)}{\alpha + \beta_p Y_0} \left[Y_0 \left\{ \alpha \mathrm{N}(d_1) - \beta_p K \mathrm{N}(d_2 + \sigma_Y^I \sqrt{S}) \right\} - \alpha K \mathrm{N}(d_2) \right. \\ & \left. + \beta_p Y_0^2 \mathbf{e}^{\sigma_Y^K \sigma_Y^I S} \mathrm{N}(d_1 + \sigma_Y^I \sqrt{S}) \right], \end{aligned} \tag{50}$$

with

$$\begin{aligned} d_1 &= \frac{\ln(\frac{Y_0}{K}) + \frac{1}{2}(\sigma_Y^K)^2 S}{\sigma_Y^K \sqrt{S}} \\ d_2 &= \frac{\ln(\frac{Y_0}{K}) - \frac{1}{2}(\sigma_Y^K)^2 S}{\sigma_Y^K \sqrt{S}} \end{aligned}$$

Applied to formula (49) we get

$$\begin{aligned} & B(0,p) \mathbf{E}_{\mathbf{Q}_p} \max(\varepsilon(Y_S - K), 0) \\ = & \frac{B(0,p)}{\alpha + \beta Y_0} \left[\varepsilon \left\{ (Y_0 - K)(\alpha + \beta Y_0) + \beta \sigma_{Y,n}^I \sigma_{Y,n}^K S \right\} \mathrm{N}(\varepsilon d) \right. \\ & \left. + \sigma_{Y,n}^K \sqrt{S} (\alpha + \beta Y_0) \mathrm{n}(\varepsilon d) \right], \end{aligned} \tag{51}$$

with

$$d = \frac{Y_0 - K}{\sigma_{Y,n}^K \sqrt{S}}.$$

4.3 Accuracy study and examples

In this section we study the accuracy of the lognormal volatility approximations derived in Section 4.1 thereby also giving some numerical examples on the size of the volatility adjustment. We compare the prices of caps and binaries on Libor-in-arrears calculated from the approximating lognormal model with the results of the exact but more involved evaluation (48). It turns out that the approximation proposed delivers results of high accuracy.

The following table shows some numerical results for the price (in basis points) of a caplet[3]

$$\Delta\ \mathbf{E}_{\mathbf{Q}_S} \max(L(S,T) - K, 0)$$

for a forward rate $L^0(S,T) = 5\%$ and various scenarios for the time S to maturity and the volatility σ. The approximate price is based on a lognormal distribution with adjusted volatility according to (45) and adjusted forward rates $\mathbf{E}_{\mathbf{Q}_S} L(S,T)$ from (14).

S	10			S	10		
σ	20,00%			σ	40,00%		
σ^*	20,14%			σ^*	43,27%		
$\mathbf{E}_{\mathbf{Q}_S} L(S,T)$	5,061%			$\mathbf{E}_{\mathbf{Q}_S} L(S,T)$	5,490%		
K	approx.	exact	% error	K	approx.	exact	% error
1%	406,334	406,327	0,000000%	1%	462,996	460,630	0,000051%
2%	312,399	312,327	0,000002%	2%	402,164	396,910	0,000132%
3%	233,987	233,839	0,000006%	3%	356,308	349,029	0,000209%
4%	173,590	173,409	0,000010%	4%	320,070	311,471	0,000276%
5%	128,767	128,595	0,000013%	5%	290,500	281,079	0,000335%
6%	95,965	95,825	0,000015%	6%	265,802	255,904	0,000387%
7%	72,021	71,920	0,000014%	7%	244,804	234,668	0,000432%
8%	54,484	54,422	0,000011%	8%	226,697	216,492	0,000471%
9%	41,560	41,531	0,000007%	9%	210,902	200,747	0,000506%
10%	31,965	31,962	0,000001%	10%	196,992	186,968	0,000536%

S	20			S	20		
σ	20,000%			σ	40,000%		
σ^*	20,42%			σ^*	50,23%		
$\mathbf{E}_{\mathbf{Q}_S} L(S,T)$	5,15%			$\mathbf{E}_{\mathbf{Q}_S} L(S,T)$	7,92%		
K	approx.	exact	% error	K	approx.	exact	% error
1%	417,952	417,780	0,000004%	1%	733,461	722,494	0,000152%
2%	336,494	335,867	0,000019%	2%	697,997	679,488	0,000272%
3%	273,187	272,205	0,000036%	3%	670,784	647,240	0,000364%
4%	224,353	223,187	0,000052%	4%	648,358	621,272	0,000436%
5%	186,347	185,128	0,000066%	5%	629,158	599,497	0,000495%
6%	156,394	155,205	0,000077%	6%	612,312	580,746	0,000544%
7%	132,483	131,374	0,000084%	7%	597,275	564,288	0,000585%
8%	113,165	112,161	0,000090%	8%	583,681	549,633	0,000619%
9%	97,386	96,497	0,000092%	9%	571,267	536,433	0,000649%
10%	84,369	83,596	0,000093%	10%	559,839	524,435	0,000675%

Errors are shown here as percentage errors of time value.

The differences between the true distribution and the approximating lognormal distribution become more transparent when comparing binary options, i.e. options with payout $\mathbf{1}_{\{L(S,T)>K\}}$ at time S. Here are some numerical comparisons:

[3] This is the price of the caplet up to a multiplication with the discount factor $B(0,S)$.

S	10			S	10		
σ	20,00%			σ	40,00%		
σ^*	20,14%			σ^*	43,27%		
$\mathbf{E}_{\mathbf{Q}_S} L(S,T)$	5,061%			$\mathbf{E}_{\mathbf{Q}_S} L(S,T)$	5,490%		
K	approx.	exact	% error	K	approx.	exact	% error
1%	9870	9873	0,03%	1%	7123	7447	4,34%
2%	8727	8735	0,10%	2%	5214	5459	4,49%
3%	6924	6929	0,08%	3%	4042	4205	3,89%
4%	5203	5204	0,02%	4%	3253	3357	3,10%
5%	3823	3821	-0,06%	5%	2690	2753	2,28%
6%	2790	2787	-0,14%	6%	2269	2303	1,49%
7%	2039	2035	-0,20%	7%	1944	1958	0,73%
8%	1498	1494	-0,24%	8%	1687	1687	0,02%
9%	1108	1105	-0,27%	9%	1479	1470	-0,65%
10%	826	824	-0,28%	10%	1308	1292	-1,27%

S	20			S	20		
σ	20,00%			σ	40,00%		
σ^*	20,42%			σ^*	50,23%		
$\mathbf{E}_{\mathbf{Q}_S} L(S,T)$	5,152%			$\mathbf{E}_{\mathbf{Q}_S} L(S,T)$	7,916%		
K	approx.	exact	% error	K	approx.	exact	% error
1%	9097	9137	0,44%	1%	4198	5135	18,25%
2%	7190	7233	0,60%	2%	3047	3653	16,57%
3%	5540	5567	0,48%	3%	2447	2864	14,57%
4%	4288	4299	0,26%	4%	2063	2362	12,68%
5%	3359	3359	0,01%	5%	1791	2011	10,94%
6%	2665	2659	-0,23%	6%	1587	1751	9,36%
7%	2141	2132	-0,45%	7%	1426	1549	7,90%
8%	1740	1729	-0,65%	8%	1297	1388	6,57%
9%	1429	1417	-0,82%	9%	1189	1256	5,33%
10%	1184	1173	-0,97%	10%	1099	1147	4,18%

4.4 Options to exchange interest rates

Consider two interest rates Y_1 and Y_2 which are set (fixed) at times S_1 and S_2, respectively. For example, Y_1 and Y_2 could be Libor rates $L(S_1, T_1)$ and $L(S_2, T_2)$ referring to different fixing dates S_1, S_2, e.g. Libor and Libor-in-arrears. One could also think of two CMS rates to be set at the same date but with different tenors.
We are interested in an option to exchange both interest rates

$$\max(Y_2 - Y_1, 0) \tag{52}$$

with payout to be paid at time $p \geq \max(S_1, S_2)$. notation simple let us suppose that $S_2 \geq S_1$.

We assume that both interest rates are lognormal under $\mathbf{Q}_p$

$$\begin{aligned} Y_i &= Y_i^0 \exp(\sigma_i W_{S_i}^i - \sigma_i^2 S_i/2) \qquad (53)\\ Y_i^0 &= \mathbf{E}_{\mathbf{Q}_p}(Y_i). \qquad (54) \end{aligned}$$

with $\mathbf{E}_{\mathbf{Q}_p}(Y_i)$ given by (19), (28) or (14), (31). According to our analysis in Section 4.1 the assumption of log-normality is at least approximately satisfied under the market model if the market volatility is adjusted according to (45).

Proposition 9 *Let the driving Brownian motions W^1 and W^2 be correlated with dynamic correlation ρ, i.e.,*

$$\mathbf{E}_{\mathbf{Q}_p}(W_t^1 W_t^2) = \rho\, t, \quad t \geq 0.$$

The fair price of the exchange option is then given by

$$B(0,p)\left[Y_2^0 \mathrm{N}(b_1) - Y_1^0 \mathrm{N}(b_2)\right] \tag{55}$$

with

$$b_{1,2} = \frac{\ln(\frac{Y_2^0}{Y_1^0}) + \frac{1}{2}(\sigma_1^2 S_1 + \sigma_2^2 S_2 \pm 2\sigma_1\sigma_2\rho S)}{\sqrt{\sigma_1^2 S_1 + \sigma_2^2 S_2 - 2\sigma_1\sigma_2\rho S}},$$

where

$$S = \min(S_1, S_2).$$

Proof. The price of the exchange option is given by

$$B(0,p)\mathbf{E}_{\mathbf{Q}_p} \max(Y_2 - Y_1, 0).$$

The calculation of this expectation is rather standard. To keep the notation simple let us suppose that $S_2 \geq S_1$. We represent $W^1_{S_1}, W^2_{S_2}$ via independent standard Gaussian random variables ξ_1, ξ_2

$$\begin{aligned} W^1_{S_1} &= \sqrt{S_1}\xi_1 \\ W^2_{S_2} &= \sqrt{S_2}(\lambda\xi_1 + \sqrt{1-\lambda^2}\xi_2) \\ \lambda &= \rho\sqrt{\frac{S_1}{S_2}}. \end{aligned}$$

First taking the expectation w.r.t. ξ_2 get us to a Black & Scholes type expression

$$\mathbf{E}_{\mathbf{Q}_p} \max(Y_2 - Y_1, 0) = \mathbf{E}_{\mathbf{Q}_p}(X_2 \mathrm{N}(d_1) - X_1 \mathrm{N}(d_2)) \tag{56}$$

with

$$\begin{aligned} X_2 &= Y_2^0 \exp(\sigma_2\sqrt{S_2}\lambda\xi_1 - \frac{1}{2}\sigma_2^2 S_2 \lambda^2) \\ X_1 &= Y_1^0 \exp(\sigma_1\sqrt{S_1}\xi_1 - \frac{1}{2}\sigma_1^2 S_1) \\ d_{1,2} &= \frac{\ln\left(\frac{X_2}{X_1}\right) \pm \frac{1}{2}\sigma_2^2 S_2(1-\lambda^2)}{\sqrt{\sigma_2^2 S_2(1-\lambda^2)}}. \end{aligned}$$

The expectation on the right hand side of (56) is now further explored integrating w.r.t. ξ_1 and applying the following well-known formula for integrals w.r.t. the standard normal density $\varphi = \mathrm{N}'$

$$\int_{-\infty}^{\infty} \mathrm{N}(ax+b)\exp(cx)\varphi(x)\mathrm{d}x = \mathrm{N}\left(\frac{ac+b}{\sqrt{1+a^2}}\right)\exp(c^2/2).$$

◊

Remark. As expected the formula (55) is related to the well-known Margrabe formula (see e.g. Hull (2009)) on the difference of two assets. Since in our case the two underlying quantities Y_2 and Y_1 are not necessarily set at the same point in time one has to adopt the inputs to Margrabes formula appropriately to take these effects into account and derive our formula (55) from Margrabes formula.

4.5 Spread options

Spread options are very popular options to cap or floor payments made from the spread of two indices, for example the spread between a long and a short interest rate. The exchange option above can be seen as a special case of the spread option

with strike 0. We use the notations of the section above. Let K be the strike of the spread options, then the payout is

$$\max(\varepsilon(Y_2 - Y_1 - K), 0)$$

with $\varepsilon = 1$ for a cap and -1 for a floor. The payout is made at time p. Note, that K can be negative.

Proposition 10 *Under the assumptions of Proposition 9, the fair price of the spread option is given for $K \geq 0$ by*

$$B(0,p) \int \varepsilon \left[X_2(\xi) N(\varepsilon d_1(\xi)) - (X_1(\xi) + K) N(\varepsilon d_2(\xi)) \right] \varphi(\xi) \mathrm{d}\xi \tag{57}$$

with

$$\begin{aligned}
X_1(\xi) &= Y_1^0 \exp(\sigma_1 \sqrt{S_1} \xi - \frac{1}{2} \sigma_1^2 S_1) \\
X_2(\xi) &= Y_2^0 \exp(\sigma_2 \sqrt{S_2} \lambda \xi - \frac{1}{2} \sigma_2^2 S_2 \lambda^2) \\
d_{1,2}(\xi) &= \frac{\ln\left(\frac{X_2(\xi)}{X_1(\xi)+K}\right) \pm \frac{1}{2}\sigma_2^2 S_2 (1-\lambda^2)}{\sqrt{\sigma_2^2 S_2 (1-\lambda^2)}} \\
\lambda &= \rho \sqrt{\frac{S_1}{S_2}}.
\end{aligned}$$

The case $K < 0$ has to treated with some care because of a potential singularity in the logarithm. In this case we use the relation

$$\max(\varepsilon(Y_2 - Y_1 - K), 0) = \max(-\varepsilon(Y_1 - Y_2 + K), 0)$$

and price the RHS using (57).

The proof is analogous to the first integration in the proof of Proposition 9, the second integration is done numerically.

4.6 Extension to general options on two indices

The spread option pricing formula can be easily generalized to price an option on the arbitrary weighted sum of two indices. Let ω_1 and ω_2 be real numbers. Then we want to price

$$\max(\varepsilon(\omega_1 Y_1 + \omega_2 Y_2 - K), 0)$$

using the same setup as in the above sections.

Proposition 11 *Let $\omega_2 > 0$. Then we have*

$$\mathbf{E}_{\mathbf{Q}_p}\left[\max(\varepsilon(\omega_1 Y_1 + \omega_2 Y_2 - K, 0))\right] = \tag{58}$$
$$\int \varepsilon\left[(\omega_1 X_1(\xi) - K) N(\varepsilon d_2^{\omega_1,\omega_2}(\xi)) + \omega_2 X_2(\xi) N(\varepsilon d_1^{\omega_1,\omega_2}(\xi))\right] \varphi(\xi) \mathrm{d}\xi$$

for $\omega_1 X_1(\xi)$ with

$$\begin{aligned}
X_1(\xi) &= Y_1^0 \exp(\sigma_1 \sqrt{S_1}\xi - \frac{1}{2}\sigma_1^2 S_1) \\
X_2(\xi) &= Y_2^0 \exp(\sigma_2 \sqrt{S_2}\lambda\xi - \frac{1}{2}\sigma_2^2 S_2 \lambda^2) \\
d_{1,2}^{\omega_1,\omega_2}(\xi) &= \begin{cases} K - \omega_1 X_1(\xi)) \geq 0 : \frac{\ln\left(\frac{\omega_2 X_2(\xi)}{K - \omega_1 X_1(\xi)}\right) \pm \frac{1}{2}\sigma_2^2 S_2(1-\lambda^2)}{\sqrt{\sigma_2^2 S_2 (1-\lambda^2)}} \\ \text{else} \qquad : \infty \end{cases} \\
\lambda &= \rho\sqrt{\frac{S_1}{S_2}}
\end{aligned}$$

where we use $N(+\infty) = 1$ and $N(-\infty) = 0$. If $\omega_2 < 0$ we replace ω_1 by $-\omega_1$, ε by $-\varepsilon$ and K by $-K$ and apply (58) to these parameters.

The spread option is given as the special case $\omega_1 = -1$ and $\omega_2 = 1$. Another interesting application is the average of two indices, i.e. $\omega_1 = \omega_2 = 0.5$.

Proof. We set

$$Y_1 = Y_1^0 \exp(\sigma_1\sqrt{S_1}\xi_1 - \frac{\sigma_1^2}{2}S_1) =: Y_1(\xi_1)$$

and

$$Y_2 = Y_2^0 \exp(\sigma_2\sqrt{S_2}\lambda\xi_1 + \sigma_2\sqrt{1-\lambda^2}\sqrt{S_2}\xi_2) =: Y_2(\xi_1)\exp(\gamma\xi_2 - \frac{1}{2}\gamma^2)$$

with $\gamma = \sigma_2\sqrt{S_2(1-\lambda^2)}$ and ξ_1, ξ_2 normal distributed and i.i.d. We do the integration over ξ_2 first:

$$\begin{aligned}
&\int_{-\infty}^{\infty} [\varepsilon(\omega_1 Y_1 + \omega_2 Y_2 - K)]_+ \varphi(\xi_2) d\xi_2 \\
= &\int_{-\infty}^{\infty} \left[\varepsilon(\omega_1 Y_1(\xi_1) + \omega_2 Y_2(\xi_1)\exp(\gamma\xi_2 - \frac{1}{2}\gamma^2) - K)\right]_+ \varphi(\xi_2) d\xi_2
\end{aligned}$$

To fix the integration limits for ξ_2, we have to consider several cases:

1. $\varepsilon = +1$

   (a) $K - \omega_1 Y_1(\xi_1) \geq 0 \quad \Rightarrow \quad \xi_2 \geq -d_2^{\omega_1,\omega_2}(\xi_1)$

   (b) $K - \omega_1 Y_1(\xi_1) < 0 \quad \Rightarrow \quad \xi_2$ arbitrary.

2. $\varepsilon = -1$

   (a) $K - \omega_1 Y_1(\xi_1) \geq 0 \quad \Rightarrow \quad \xi_2 \leq -d_2^{\omega_1,\omega_2}(\xi_1)$

   (b) $K - \omega_1 Y_1(\xi_1) < 0 \quad \Rightarrow \quad$ No solution for ξ_2.

In case 1(a) we get

$$\begin{aligned}
&= \int_{-d_2(\xi_1)}^{\infty} \left(\omega_1 Y_1(\xi_1) + \omega_2 Y_2(\xi_1) \exp(\gamma \xi_2 - \frac{1}{2}\gamma^2) - K \right) \varphi(\xi_2) d\xi_2 \\
&= (\omega_1 Y_1(\xi_1) - K) N(d_2^{\omega_1,\omega_2}(\xi_1)) + \omega_2 Y_2(\varepsilon \xi_1) N(d_1^{\omega_1,\omega_2}(\xi_1)).
\end{aligned}$$

The case 1(b) yields

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \left(\omega_1 Y_1(\xi_1) + \omega_2 Y_2(\xi_1) \exp(\gamma \xi_2 - \frac{1}{2}\gamma^2) - K \right) \varphi(\xi_2) d\xi_2 \\
&= \omega_1 Y_1(\xi_1) - K + \omega_2 Y_2(\xi_1).
\end{aligned}$$

The case 2(a) gives

$$\begin{aligned}
&= \int_{-\infty}^{-d_2(\xi_1)} \left(-\omega_1 Y_1(\xi_1) - \omega_2 Y_2(\xi_1) \exp(\gamma \xi_2 - \frac{1}{2}\gamma^2) + K \right) \varphi(\xi_2) d\xi_2 \\
&= (K - \omega_1 Y_1(\xi_1)) N(-d_2^{\omega_1,\omega_2}(\xi_1)) - \omega_2 Y_2(\varepsilon \xi_1) N(-d_1^{\omega_1,\omega_2}(\xi_1)) \\
&= \varepsilon \left[(\omega_1 Y_1(\xi_1) - K) N(\varepsilon d_2^{\omega_1,\omega_2}(\xi_1)) + \omega_2 Y_2(\xi_1) N(\varepsilon d_1^{\omega_1,\omega_2}(\xi_1)) \right].
\end{aligned}$$

□

4.7 Spread options in the Bachelier case

Here we assume instead of (53) a normal distribution under $\mathbf{Q}_p$ with volatility $\sigma_{i,n}$,

$$Y_i = Y_i^0 + \sigma_{i,n} W_{S_i}^i, \tag{59}$$

with $\mathbf{E}_{\mathbf{Q}_p}(Y_i)$ given by (21) or (16). Analogously to the log-normal case the volatility $\sigma_{i,n}$ may be adjusted using (47). Pricing of spread options in the normal case is much simpler than the log-normal case, since the sum of two normal distributed variables is again normal distributed. Therefore we easily obtain for the price of an option on the weighted sum of two indices:

Proposition 12 *Let the driving Brownian motions W^1 and W^2 be correlated with dynamic correlation ρ_n, i.e.,*

$$\mathbf{E}_{\mathbf{Q}_p}(W_t^1\, W_t^2) = \rho_n\, t, \;\; t \geq 0.$$

We set

$$\begin{aligned} \bar{Y}_0 &= \omega_1 Y_0^1 + \omega_2 Y_0^2 && (60)\\ \bar{S} &= \min(S_1, S_2) && (61)\\ \bar{\sigma}^2 \bar{S} &= \omega_1^2 \sigma_{1,n}^2 S_1 + \omega_2^2 \sigma_{2,n}^2 S_2 + 2\omega_1\omega_2\sigma_{1,n}\sigma_{2,n}\rho_n \bar{S} && (62)\\ d &= \frac{\bar{Y}_0 - K}{\bar{\sigma}\sqrt{\bar{S}}} && (63) \end{aligned}$$

Then we get for the price of the weighted spread option

$$\mathbf{E}_{\mathbf{Q}_p}\left[\max(\varepsilon(\omega_1 Y_1 + \omega_2 Y_2 - K, 0))\right] = \varepsilon(\bar{Y}_0 - K)\mathrm{N}(\varepsilon d) + \bar{\sigma}\sqrt{\bar{S}}\mathrm{n}(\varepsilon d) \quad (64)$$

This is just the Bachelier formula applied to $\bar{Y}_0$ with the mixture volatility $\bar{\sigma}$.

5 Variable interest rates in foreign currency

Consider a domestic floating interest rate Y_S^d which is set (fixed) in the market at time S. Examples of interest are $Y_S^d = L(S,T)$, the Libor for the interval $[S,T]$, and, $Y_S^d = X$ with $S = T_0$ and X the swap rate with reference dates $T_0 < T_1 < \cdots < T_n$.
We are interested in the price of the rate Y_S^d to be paid in foreign currency units at some time $p \geq S$.

5.1 General relationships

Let N^c denote a numeraire process with associated martingale measure $\mathbf{Q}^c$ for the domestic ($c = d$) and foreign ($c = f$) economy. The foreign exchange rate X_t at time t is the value of one unit foreign in domestic currency at time t. Any foreign asset S_t^f is considered to be a traded asset in the domestic economy if multiplied with the exchange rate: $S_t^f \cdot X_t$.
The value today of a domestic payoff Z_T^d to be paid in foreign units and at time T is by the general theory

$$N_0^f \mathbf{E}_{\mathbf{Q}^f} \frac{Z_T^d}{N_T^f}.$$

On the other hand, the same payoff translated back into domestic currency with the exchange rate at time T should trade at the same price, therefore

$$N_0^f \mathbf{E}_{\mathbf{Q}^f} \frac{Z_T^d}{N_T^f} = \frac{N_0^d}{X_0} \mathbf{E}_{\mathbf{Q}^d} \frac{Z_T^d X_T}{N_T^d}.$$

Consequently, the density of the two martingale measures $\mathbf{Q}^f$ and $\mathbf{Q}^d$ on the information structure up to time T is given by

$$\begin{aligned} \frac{\mathrm{d}\mathbf{Q}^d}{\mathrm{d}\mathbf{Q}^f} &= \frac{N_T^d}{X_T N_T^f} \frac{X_0 N_0^f}{N_0^d} \\ \frac{\mathrm{d}\mathbf{Q}^f}{\mathrm{d}\mathbf{Q}^d} &= \frac{X_T N_T^f}{N_T^d} \frac{N_0^d}{X_0 N_0^f}. \end{aligned} \tag{65}$$

This implies the following

Corollary 2 *The process* $(\frac{X_t N_t^f}{N_t^d})_{t \geq 0}$ *is a* $\mathbf{Q}^d$ *martingale.*

5.2 Quanto adjustments

Now we come back to the valuation of a variable interest rate Y_S^d paid at time p in foreign units. Using the time p maturity foreign zero bond $B^f(.,p)$ as numeraire together with the foreign time p forward measure and relation (65) as well as Corollary 2 the price is

$$\begin{aligned} B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f} \frac{Y_S^d}{B^f(p,p)} &= \frac{N_0^d}{X_0} \mathbf{E}_{\mathbf{Q}^d} \left(Y_S^d \frac{X_S B^f(S,p)}{N_S^d} \right) \\ &= \frac{N_0^d}{X_0} \mathbf{E}_{\mathbf{Q}^d} \left(Y_S^d \frac{X_S B^f(S,p)}{B^d(S,p)} \frac{B^d(S,p)}{N_S^d} \right). \end{aligned} \tag{66}$$

By definition the ratio $\frac{B^d(S,p)}{N_S^d}$ is a $\mathbf{Q}^d$ martingale in the time variable $S \leq p$. The expression $\frac{X_S B^f(S,p)}{B^d(S,p)}$ is the time S forward foreign exchange rate for delivery at time $p \geq S$.

Our goal is to make (66) more explicit in terms of market observable quantities like forward rates, volatilities etc. To this end we need to impose some modeling assumptions. From now on we assume that our domestic numeraire N^d is the natural numeraire associated with the interest rate Y_S^d, i.e., if $Y_S^d = L(S,T)$ then $N^d = B^d(.,T)$ and $\mathbf{Q}^d = \mathbf{Q}_T^d$ or if $Y_S^d = X$ then $N^d = \sum \Delta_i B^d(.,T_i)$ and $\mathbf{Q}^d = \mathbf{Q}_{\text{Swap}}^d$.

Under the Black model, the distribution of Y_S^d under the measure $\mathbf{Q}^d$ is log-normal

$$Y_S^d = Y_0^d \exp(\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S), \tag{67}$$

whereas under the Bachelier model the distribution of Y_S^d is normal,

$$Y_S^d = Y_0^d + \sigma_{Y,n} W_S, \tag{68}$$

with

$$Y_0^d = \mathbf{E}_{\mathbf{Q}^d} Y_S^d.$$

The rate Y_0^d is the forward Libor $L^0(S,T)$ (cf. (9)) or the forward swap rate X^0 (cf. (3)), respectively, as of today.

For the last term on the right hand side of equation (66) we suppose, as we have already done in previous sections (cf. e.g. equation (43)), a linear model of the form

$$\frac{B^d(S,p)}{N_S^d} = \alpha + \beta_p Y_S^d \tag{69}$$

with α and β_p given by (18) or (27) and (26), respectively.

The most critical assumption we are going to impose is on the distribution of the forward foreign exchange rate. We suppose a lognormal distribution under $\mathbf{Q}^d$

$$X_S \frac{B^f(S,p)}{B^d(S,p)} = X_0^\star \exp(\sigma_{\text{fx}} W_S^{\text{fx}} - \frac{1}{2}\sigma_{\text{fx}}^2 S) \tag{70}$$

with expectation

$$X_0^\star = \mathbf{E}_{\mathbf{Q}^d}\left(X_S \frac{B^f(S,p)}{B^d(S,p)}\right)$$

to be calculated below. Observe that the assumption of log-normality of the forward foreign exchange rate is in general not compatible with the assumption of a lognomal rate Y_S^d. In practice, we will identify the volatility σ_{fx} with the implied volatility of a foreign exchange rate option with maturity S. This would be a crucial simplification as long as the payment date p is not close to S, which is, however, the case in most applications.

To calculate the expectation $X_0^\star$ we make use of the fact that according to Corollary 2 ($\frac{X_S B^f(S,p)}{N_S^d}$) is a $\mathbf{Q}^d$ martingale in $S \le p$. Together with (69) and (67) this yields

$$\begin{aligned} &\frac{X_0 B^f(0,p)}{N_0^d} \\ &= \mathbf{E}_{\mathbf{Q}^d}\left(\frac{X_S B^f(S,p)}{N_S^d}\right) = \mathbf{E}_{\mathbf{Q}^d}\left(X_s \frac{B^f(S,p)}{B^d(S,p)} \frac{B^d(S,p)}{N_S^d}\right) \\ &= \begin{cases} X_0^\star(\alpha + \beta_p Y_0^d \exp(\rho\sigma_{\text{fx}}\sigma_Y S)) & : \quad \text{in case of (67)}, \\ X_0^\star(\alpha + \beta_p Y_0^d + \beta_p \rho \sigma_{\text{fx}} \sigma_{Y,n} S) & : \quad \text{in case of (68)}, \end{cases} \end{aligned}$$

with ρ as correlation between the driving Brownian motions W^{fx} and W. As a consequence the expectation of the forward foreign exchange rate under $\mathbf{Q}^d$ is

$$X_0^\star = \frac{X_0 B^f(0,p)}{N_0^d(\alpha + \beta_p Y_0^d \exp(\rho \sigma_{\text{fx}} \sigma_Y S))},$$

for the model (67), and

$$X_0^\star = \frac{X_0 B^f(0,p)}{N_0^d(\alpha + \beta_p Y_0^d + \beta \rho \sigma_{\text{fx}} \sigma_{Y,n} S)},$$

in case of the model (68). Now we are ready to follow up with an explicit calculation of (66). Substituting (67), (69) and (70) we obtain

$$\begin{aligned}
& B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f} \frac{Y_S^d}{B^f(p,p)} \\
&= \frac{N_0^d}{X_0} Y_0^d \frac{X_0 B^f(0,p)}{N_0^d(\alpha + \beta_p Y_0^d \exp(\rho \sigma_{\text{fx}} \sigma_Y S))} \mathbf{E}_{\mathbf{Q}^d} \Big(\exp(\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S) \\
&\quad \exp(\sigma_{\text{fx}} W_S^{\text{fx}} - \frac{1}{2}\sigma_{\text{fx}}^2 S)(\alpha + \beta_p Y_0^d \exp(\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S)) \Big) \\
&= B^f(0,p) Y_0^d \frac{\exp(\rho \sigma_{\text{fx}} \sigma_Y S)(\alpha + \beta_p Y_0^d \exp(\rho \sigma_{\text{fx}} \sigma_Y S + \sigma_Y^2 S))}{\alpha + \beta_p Y_0^d \exp(\rho \sigma_{\text{fx}} \sigma_Y S)}.
\end{aligned}$$

Proposition 13 *Under the conditions (67), (69) and (70) the quanto adjusted forward rate for a payment of the variable rate Y_S^d set at time S and paid at time $p \geq S$ in foreign currency units is given by*

$$\boxed{\mathbf{E}_{\mathbf{Q}_p^f} Y_S^d = \bar{Y}_0^d \frac{(\alpha + \beta_p \bar{Y}_0^d \exp(\sigma_Y^2 S))}{\alpha + \beta_p \bar{Y}_0^d}}, \tag{71}$$

where $\bar{Y}_0^d = Y_0^d \exp(\rho \sigma_{\text{fx}} \sigma_Y S)$ and with α and β_p as in (18) or (27) and (26), respectively, and ρ as correlation between the driving Brownian motions.

Analogously, we obtain the following result in case of the Bachelier model.

Proposition 14 *Under the conditions (68), (69) and (70) the quanto adjusted forward rate for a payment of the variable rate Y_S^d set at time S and paid at time $p \geq S$ in foreign currency units is given by*

$$\boxed{\mathbf{E}_{\mathbf{Q}_p^f} Y_S^d = \bar{Y}_0^d \frac{(\alpha + \beta_p \bar{Y}_0^d) + S \beta_p \sigma_{Y,n}^2 / \bar{Y}_0^d}{\alpha + \beta_p \bar{Y}_0^d}}, \tag{72}$$

where $\bar{Y}_0^d = Y_0^d + \rho \sigma_{\text{fx}} \sigma_{Y,n} S$ and with α and β_p as in (18) or (27) and (26), respectively, and ρ as correlation between the driving Brownian motions.

Remark. In the special case of the foreign unit being the domestic unit, i.e. $\sigma_{\text{fx}} = 0$, formula (71) reduces to (20), (21) or (31) as expected.

Example. To illustrate the impact and size of the quanto adjustment consider as an example a diff swap, i.e. $Y_S^d = L(S,T)$ to be paid at time $p = T$ in foreign currency units. In this particular case we have $\alpha = 1$ and $\beta_p = \beta_T = 0$ and the adjustment reduces to

$$\mathbf{E}_{\mathbf{Q}_T^f} L(S,T) = L^0(S,T)\,\exp(\rho\sigma_{\text{fx}}\sigma_Y S).$$

For $S = 5, \sigma_{\text{fx}} = 15\%, \sigma_Y = 18\%$ and $\rho = 50\%$ this gives an adjustment factor on the forward rate of

$$\exp(\rho\sigma_{\text{fx}}\sigma_Y S) = 1,0698.$$

5.3 Quantoed options on interest rates

In this section we extend the analysis of the previous section to standard options on a domestic interest rate Y_S^d with payment at an arbitrary time $p \geq S$ but in foreign currency units. This can be seen also as an extension of the result in Section 4.2. We use the notation and assumptions (67) resp. (68), (69) and (70) as in the previous section.

Consider a call option on the domestic rate Y_S^d with strike K paid at $p \geq S$ in foreign currency. By the general theory the price of this option is (compare also (66))

$$\begin{aligned} & B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f}\frac{\max(Y_S^d - K, 0)}{B^f(p,p)} \\ &= \frac{N_0^d}{X_0}\mathbf{E}_{\mathbf{Q}^d}\left(\max(Y_S^d - K,0)\frac{X_S B^f(S,p)}{B^d(S,p)}\frac{B^d(S,p)}{N_S^d}\right) \\ &= \frac{N_0^d}{X_0}\mathbf{E}_{\mathbf{Q}^d}\left(\max(Y_S^d - K,0)\frac{X_S B^f(S,p)}{B^d(S,p)}(\alpha + \beta_p Y_S^d)\right). \qquad (73) \end{aligned}$$

This expectation can be calculated again explicitly under the given modeling assumptions and we obtain the following results.

Proposition 15 *Under the conditions (67), (69) and (70) the call resp. put option on the domestic variable rate Y_S^d set at time S and paid at time $p \geq S$ in foreign*

currency units is given by

$$
\begin{aligned}
& B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f} \max(Y_S^d - K, 0) \qquad (74)\\
&= \frac{B^f(0,p)}{\alpha + \beta_p \bar{Y}_0^d}\varepsilon \Big[\bar{Y}_0^d \mathrm{N}(\varepsilon d_1)(\alpha - \beta_p K) - \alpha K \mathrm{N}(\varepsilon d_2) \\
&\qquad + \beta_p (\bar{Y}_0^d)^2 \mathbf{e}^{\sigma_Y^2 S} \mathrm{N}(\varepsilon(d_1 + \sigma_Y \sqrt{S})) \Big]
\end{aligned}
$$

with

$$
\begin{aligned}
d_1 &= \frac{\ln(\frac{\bar{Y}_0^d}{K}) + \frac{1}{2}\sigma_Y^2 S}{\sigma_Y \sqrt{S}} \\
d_2 &= \frac{\ln(\frac{\bar{Y}_0^d}{K}) - \frac{1}{2}\sigma_Y^2 S}{\sigma_Y \sqrt{S}},
\end{aligned}
$$

where $\bar{Y}_0^d = Y_0^d \exp(\sigma_{\mathrm{fx}} \sigma_Y \rho S)$ *is the convexity adjusted forward (cf. (71) for* $p = T$*) and* α *and* β_p *as in (18) or (27) and (26), respectively, and* ρ *as correlation between the driving Brownian motions. Finally, the formula for a digital is*

$$
B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f} \Big(\mathbf{1}_{\{Y_S^d > K\}} \Big) = \frac{B^f(0,p)}{\alpha + \beta_p \bar{Y}_0^d} \Big[\beta_p \bar{Y}_0^d \mathrm{N}(d_1) + \alpha \mathrm{N}(d_2) \Big]. \qquad (75)
$$

Remark 2 *As in Remark 1, the implementation distinguishes two volatilities in the evaluation of formula (73): the index volatility* σ_Y^I *used for the convexity term* $(\alpha + \beta_p Y_S^d)$ *and the strike volatility* σ_Y^K *that is used for the payoff* $\max(Y_S^d - K, 0)$*. The corresponding formula is*

$$
\begin{aligned}
& B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f} \max(Y_S^d - K, 0) \qquad (76)\\
&= \frac{B^f(0,p)}{\alpha + \beta_p \bar{Y}_0^d}\varepsilon \Big[\alpha \Big[\bar{\bar{Y}}_0^d \mathrm{N}(\varepsilon d_1) - K \mathrm{N}(\varepsilon d_2) \Big] \\
&\qquad + \beta_p \bar{Y}_0^d \Big[\bar{\bar{Y}}_0^d \mathbf{e}^{\sigma_Y^K \sigma_Y^I S} \mathrm{N}(\varepsilon(d_1 + \sigma_Y^I \sqrt{S})) - K \mathrm{N}(\varepsilon(d_2 + \sigma_Y^I \sqrt{S})) \Big] \Big]
\end{aligned}
$$

with

$$
\begin{aligned}
d_1 &= \frac{\ln(\frac{\bar{\bar{Y}}_0^d}{K}) + \frac{1}{2}(\sigma_Y^K)^2 S}{\sigma_Y^K \sqrt{S}} \\
d_2 &= \frac{\ln(\frac{\bar{\bar{Y}}_0^d}{K}) - \frac{1}{2}(\sigma_Y^K)^2 S}{\sigma_Y^K \sqrt{S}},
\end{aligned}
$$

where $\bar{Y}_0^d = Y_0^d \exp(\sigma_{\text{fx}} \sigma_Y^I \rho S)$ *is the convexity adjusted forward and* $\bar{\bar{Y}}_0^d = Y_0^d \exp(\sigma_{\text{fx}} \sigma_Y^K \rho S)$.

Analogously, we obtain the following result in case of the Bachelier model.

Proposition 16 *Under the conditions (68), (69) and (70) the call resp. put option on the domestic variable rate Y_S^d set at time S and paid at time $p \geq S$ in foreign currency units is given by*

$$\begin{aligned} & B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f} \max\left(\varepsilon(Y_S^d - K), 0\right) \qquad (77) \\ &= \frac{B^f(0,p)}{\alpha + \beta_p \bar{Y}_0^d} \Bigg[\varepsilon\Big[(\bar{Y}_0^d - K)(\alpha + \beta_p \bar{Y}_0^d) + \beta_p \sigma_{Y,n}^2 S\Big] \mathrm{N}(\varepsilon d) \\ &\qquad + \Big[\sigma_{Y,n}\sqrt{S}(\alpha + \beta_p \bar{Y}_0^d)\Big] \mathrm{n}(\varepsilon d) \Bigg] \end{aligned}$$

with

$$d = \frac{\bar{Y}_0^d - K}{\sigma_{Y,n}\sqrt{S}},$$

where $\bar{Y}_0^d = Y_0^d + \sigma_{\text{fx}} \sigma_{Y,n} \rho S$ *is the convexity adjusted forward (cf. (72) for $p = T$) and α and β_p as in (18) or (27) and (26), respectively, and ρ as correlation between the driving Brownian motions. Finally, the formula for a digital is*

$$B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f}\left(\mathbf{1}_{\{Y_S^d > K\}}\right) = \frac{B^f(0,p)}{\alpha + \beta_p \bar{Y}_0^d} \Big\{ (\alpha + \beta_p \bar{Y}_0^d)\mathrm{N}(d) + \beta_p \sigma_{Y,n}\sqrt{S}\mathrm{n}(d) \Big\}. \qquad (78)$$

For the special case of $Y_S = L(S,T)$, $p = T$, the assumption (69) of a linear model is trivially satisfied and the corresponding valuation in (75) reduces to formula (11.27) in Brigo and Mercurio (2006), Section 14.4.2.

In case that the option is not quantoed, i.e., we can assume $\sigma_{\text{fx}} = 0$, formula (75) reduces to the general formula (48) for options on interest rates with arbitrary payment date.

Remark 3 *As in Remark 1, the implementation distinguishes two volatilities in the evaluation of formula (73): the index volatility $\sigma_{Y,n}^I$ used for the convexity term $(\alpha + \beta_p Y_S^d)$ and the strike volatility $\sigma_{Y,n}^K$ that is used for the payoff $\max(Y_S^d - K, 0)$. The corresponding formula is*

$$\begin{aligned} & B^f(0,p)\mathbf{E}_{\mathbf{Q}_p^f} \max\left(\varepsilon(Y_S^d - K), 0\right) \qquad (79) \\ &= \frac{B^f(0,p)}{\alpha + \beta_p \bar{Y}_0^d} \Bigg[\varepsilon\left[(\bar{\bar{Y}}_0^d - K)(\alpha + \beta_p \bar{Y}_0^d) + \beta_p \sigma_{Y,n}^K \sigma_{Y,n}^I S\right] \mathrm{N}(\varepsilon d) \\ &\qquad + \left[\sigma_{Y,n}^K \sqrt{S}(\alpha + \beta_p \bar{Y}_0^d)\right] \mathrm{n}(\varepsilon d) \Bigg] \end{aligned}$$

with

$$d = \frac{\bar{\bar{Y}}_0^d - K}{\sigma_{Y,n}^K \sqrt{S}},$$

where $\bar{Y}_0^d = Y_0^d + \sigma_{\text{fx}} \sigma_{Y,n}^I \rho S$ *is the convexity adjusted forward and* $\bar{\bar{Y}}_0^d = Y_0^d + \sigma_{\text{fx}} \sigma_{Y,n}^K \rho S$.

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