A Simple and Exact Simulation Approach to Heston Model

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Abstract

In this paper we will propose a simple approach to simulating Heston model efficiently and accurately. All existing simulation schemes so far directly work with the mean-reverting square root process of the variance in Heston model, instead we transform the variance to an equivalent volatility which follows a mean-reverting Ornstein-Uhlenbeck process. We will show it is more convenient to simulate the transformed volatility process than the original variance process since the new Ornstein-Uhlenbeck process does not have any term of square root, and is not restricted to any parameter restriction. Based on the transformed volatility process, we suggest a simple and exact scheme for the simulation of Heston model. Numerical examples show that the new scheme and Andersen's QE scheme perform very closely, and outperform other schemes such as log-normal scheme. While QE scheme suffers from the problem of "leaking correlation", transformed volatility scheme does not, and therefore, provides a high-quality alternative to the existing simulation schemes for Heston model.

Keywords: Simulation, Stochastic volatility, Heston model, Mean-reverting squared root process, Mean-reverting Ornstein-Uhlenbeck process, Option prices.

1 Introduction

Heston model (1993) is the most successful stochastic volatility model that attempts to capture the smile effect observed in implied volatilities of liquidly traded options, and to fulfill the gap of the unrealistic constant volatility assumed in the Black-Scholes model (1970). In Heston model, variances, not volatilities, are specified to follow a mean-reverting square root process. This is a process that is widely applied in finance, for example, the CIR short rate model (1985), and affine-structure model. More recently, mean-reverting square root process is often used as a leading driving stochastic factor to build smile models within LMM framework, see Andersen and Brotherton-Ratcliffe (2001), Wu and Zhang (2002), Piterbarg (2003) and Zhu (2007). The mean reason for a preference of square root diffusion lies in the simple analytic tractability of its characteristic functions in spite of a non-Gaussian distribution. Given analytic pricing formulas in Heston-like models, we can calibrate the corresponding models to market data efficiently and robustly.

In practical applications, however, an efficient and accurate simulation of a mean-reverting square root process should be equally important as analytic tractability and quick calibration since we will finally use the proposed model, coupled with the calibrated model parameters, to price and hedge exotic structures. Monte Carlo simulation is sometimes the only way to deliver reliable prices for non-liquidly traded options. In contrast to its nice analytic tractability, efficient and robust simulation of mean-reverting square root process is not a trivial task. Generally, we will encounter two problems in the simulation of Heston model: First one is the unfortunate parameter restriction that allows for negative variances. Second one is due to the nature of simulation that can not exclude negative variances in paths. Both problems break down the simulation. In the past one and half decade after Heston model, some discretization schemes are suggested to improve the straightforward Euler scheme, and particularly to avoid negative values in paths. Two approaches are usually employed, namely, more sophisticated discretization schemes, for example, Kahl and Jäckel(2005), Broadie and Kaya (2006), and moment matching methods, for example, log-normal scheme. Recently, Andersen (2007) provided an good survey on the simulation issues involving square root process, and proposed two schemes based on moment-matching methods. One of his schemes called QE works very well and delivers an efficient simulation for Heston model. As noticed by Andersen himself, if the distribution of a mean-reverting square root process is matched by other distribution in the sense of first and second moments, it is not clear whether the correlation between the underlying asset and the displaced distribution still preserves the true correlation. Regarding the fact that the correlation in Heston model plays a key role in generating volatility

skew, we have to be careful with the problem of "leaking correlation" in the application of QE scheme.

In this paper, we propose an alternative approach to simulating square root process. Instead of focusing on the square root process of variances, we transform the variance process to a volatility process, and obtain a new formulation of Heston model. The transformed volatility process has a structure of a mean-reverting Ornstein-Uhlenbeck process, and its simulation is rather simple. Even with a conventional Euler scheme, we can achieve efficient and accurate simulations for Heston model.

This paper is organized as follows. In Section 2 we briefly review Heston model and transform it in terms of volatility. Section 3 proposes a simple and exact simulation scheme based on the transformed volatility process. Section 4 delivers some numerical examples and compare the performances of our new scheme, QE scheme and log-normal scheme. Finally, we conclude in Section 5.

2 Reformulation of Heston Model

Heston model (1993) is the first stochastic volatility model that systematically deal with the valuation of options with CFs and provides a quasi closed-form solution for options. Heston does not model stochastic volatilities directly, but stochastic variances. Denote $V(t) = v^2(t)$ as variance, Heston model consists of the following two processes,

$$\begin{aligned} \frac{dS(t)}{S(t)} &= rdt + \sqrt{V(t)}dW_1(t), && (1) \\ dV(t) &= \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_2(t), && (2) \\ dW_1(t)dW_2(t) &= \rho dt. \end{aligned}$$

The process specifying the variance $V(t)$ is identical to the one that Cox-Ingersoll-Ross (1985) apply for short interest rate, and is called mean-reverting square root process. The mean-reversion is a desired property for stochastic volatility or variance and is well documented by many empirical studies. The parameter θ marks the long-term level that variance gradually converges to. The parameter κ controls the speed of variance's reverting to θ. The parameter σ is refereed to as volatility of variance since it scales the diffusion term of variance process. Let $F_{\chi^2}(x, d, n)$ denote a non-central chi-square distribution with d as the degrees of freedom and n as non-centrality parameter. The variance $V(t)$ conditional on $V(s)$ is dis-

tributed as a non-central chi-square distribution $F_{\chi^2}(aV(s), d, n)$ where

$$\begin{aligned} d &= \frac{4\kappa\theta}{\sigma^2}, \\ n &= \frac{4\kappa e^{-\kappa(t-s)}}{\sigma^2(1-e^{-\kappa(t-s)})}, \\ a &= \frac{n}{e^{-\kappa(t-s)}} = \frac{4\kappa}{\sigma^2(1-e^{-\kappa(t-s)})}. \end{aligned}$$

Based on the properties of a non-central chi-square distribution, $V(t)$ has the following two conditional moments

$$\mathbf{E}[V(t)|V(s)] = \theta + (V(s) - \theta)e^{-\kappa(t-s)}, \tag{3}$$

$$\mathbf{Var}[V(t)|V(s)] = V(s)\frac{\sigma^2}{\kappa}e^{-\kappa(t-s)}\left[1-e^{\kappa(t-s)}\right] + \frac{\sigma^2\theta}{2\kappa}\left[1-e^{\kappa(t-s)}\right]^2. \tag{4}$$

If κ, θ and σ satisfy the following condition

$$2\kappa\theta > \sigma^2, \quad V_0 > 0, \tag{5}$$

it can be shown that variances $V(t)$ are always positive and the variance process given in (1) is then well-defined under the above condition. However, unconstrained calibration of Heston model does not ensure the above condition and leads to a prior possible negative values of $V(t)$ in simulation. This makes the calculation of $\sqrt{V(t)\Delta}$ impossible and breaks down simulation. Therefore, the constrained calibration of Heston model with an embedded parameter restriction is necessary, and however, sacrifices some fitting quality.

But even the required parameter restriction is satisfied in Heston model, it is still possible to generate some negative values of $V(t_h)$ in practical simulations. To see this, we consider the process behavior in a discrete time step $[t, t+\Delta]$,

$$V(t+\Delta) = V(t) + \kappa[\theta - V(t)]\Delta + \sigma\sqrt{V(t)\Delta}Z(t),$$

where $Z(t)$ is a Gaussian random variable at time t. If $Z(t)$ is drawn to be a large negative value, so that

$$V(t) + \kappa[\theta - V(t)]\Delta + \sigma\sqrt{V(t)\Delta}Z(t) < 0,$$

we obtain $V(t+\Delta) < 0$. This finding is sometimes more frustrating than the parameter constraints (5). This simple and naive approach to avoid computing $\sqrt{V(t+\Delta)}$ is to truncate $V(t+\Delta)$ to $\max[V(t+\Delta), 0]$. This truncation scheme is not robust and generates largely biased paths that are not applicable for pricing options.

In one word, negative variances $V(t)$ lead to a break-down of computation of $\sqrt{V(t)}$. Approximating mean-reverting square-root process via a process or a distribution without the term $\sqrt{V(t)}$ seems to be the only way to overcome the simulation problem encountered in Heston model. Modification of discretization scheme and moment-matching are two approaches applied often in financial literatures. Here we suggest a third way to eliminate square root problem and consider an equivalent volatility process in accordance with the original variance process.

Let $v(t) = \sqrt{V(t)}$ denote the stochastic volatility in Heston model. Applying Ito's Lemma, we obtain the following stochastic process for $v(t)$,

$$
\begin{aligned}
dv(t) &= \frac{1}{2}\kappa\big[(\theta - \frac{\sigma^2}{4\kappa})v^{-1}(t) - v(t)\big]dt + \frac{1}{2}\sigma dW_2(t) && (6)\\
&= \kappa_v[\theta_v - v(t)]dt + \sigma_v dW_2(t), && (7)
\end{aligned}
$$

Compared with the process of $V(t)$ given in (1), we can observe that $v(t)$ follows now a mean-reverting Ornstein-Uhlenbeck process and does not have any term of square root. The mean-reversion speed κ_v of $v(t)$ is equal to $\frac{1}{2}\kappa$, and the volatility σ_v of volatility is equal $\frac{1}{2}\sigma$, equivalent to the half values of the corresponding variance process $V(t)$.

It is worth mentioning the mean level θ_v of $v(t)$ which is equal to $(\theta - \sigma^2/(4\kappa))v^{-1}(t)$. First, the mean level θ_v is not deterministic, but stochastic. This means we can not expect the volatility will converges to a given value in Heston model. Second, θ_v has two terms $\theta v^{-1}(t)$ and $-\sigma^2/(4\kappa)v^{-1}(t)$. While the first term $\theta v^{-1}(t)$ is of a order of $v(t)$, the second term $-\sigma^2/(4\kappa)v^{-1}(t)$ could be very large. These two effects together could produces a small, even negative mean level of volatility in Heston model. The condition that θ_v keeps positive is $4\kappa\theta > \sigma^2$. Obviously, the condition (5) for positive variances $V(t)$ is sufficient for a positive mean level of volatility $v(t)$ in Heston model.

Some interesting observations with respect to the volatility process (7) are in order. First, the stochastic processes (1) and (7) constitute a stochastic volatility model that completely coincides with the Heston model, and express the dynamics of underlying in terms of volatility, not in terms of variance. Second, the stochastic volatility process (7) is always well-defined and is not restricted to any parameter constraint while the variance process (2) has only a mathematical meaning under the constraint (5). Therefore, The process (7) can be considered as a generating process of the variances $V(t)$ in Heston model, and accommodates more dynamics of volatility than the variance in the sense of parameter constraint. Next, the volatility process (7) should match the Heston option pricing formula more closely than the variance process (2) since the Heston option pricing formula is also not restricted to any parameter constraint. We can put any model parameters into the pricing formula and obtain meaningful prices. There are no reports on the failure of Heston pricing formula due to inappreciable parameters. Finally, the process of $v(t)$ follows the mean-reversion Ornstein-Uhlenbeck process with a stochastic mean level θ_v. If θ_v is reduced to be constant, we obtain here Schöbel-Zhu model (1999). It is insightful to observe that how different structures of mean level of $v(t)$ in Heston model and Schöbel-Zhu model lead to different interpretations of models and different pricing formulas.

3 Transformed Volatility Scheme

A reformation of Heston in terms of volatility, as given by (7) has opened us a total new way to re-exam Heston model. A natural step is to simulate Heston model by using the volatility process instead of the original variance process. Given an uniform time partition $\{t_0 = 0, t_1, t_2, \cdots, t_{H-1}, t_H\}$ and $X(t) = \ln S(t)$ and $\Delta = t_h - t_{h-1}$, Heston model and its reformation can be discretized according to Euler's scheme respectively:

Heston Model (Variance Form):

$$\begin{aligned} X(t_{h+1}) &= X(t_h) + [r(t_h) - \frac{1}{2}V(t_h)]\Delta + \sqrt{V(t_h)\Delta}Z_1(t_h), \\ V(t_{h+1}) &= V(t_h) + \kappa[\theta - V(t_h)]\Delta + \sigma\sqrt{V(t_h)\Delta}Z_2(t_h). \end{aligned} \tag{8}$$

Heston Model (Volatility Form):

$$\begin{aligned} X(t_{h+1}) &= X(t_h) + [r(t_h) - \frac{1}{2}v^2(t_h)]\Delta + v(t_h)\sqrt{\Delta}Z_1(t_h), \\ v(t_{h+1}) &= v(t_h) + \frac{1}{2}\kappa\big[\theta_v(t_h) - v(t_h)\big]\Delta + \frac{1}{2}\sigma\sqrt{\Delta}Z_2(t_h). \end{aligned} \tag{9}$$

Here $Z_1(t_h)$ and $Z_2(t_h)$ are two correlated standard normally distributed random variables drawn at time t_h. We call the Euler scheme of Heston model in volatility form given in (9) the transformed volatility scheme.

The advantages of the new scheme based on the transformed volatility are at least threefold. First, the transformed volatility process $v(t)$ coincides completely with the original Heston model and at the same time avoids any calculation of square root. The above transform is theoretically exact. The only error of the simulation comes from discretization and the generated random numbers. Second, any parameter constellation is allowed in the process of $v(t)$. Negative values of $v(t)$ still lead to positive variances $V(t)$, and can be similarly interpreted as in Schöbel-Zhu model. Finally, the transformed Heston model keeps the correlation of original Heston model. The Wiener process driving the stochastic volatility $v(t)$ is same as the one driving stochastic variance $V(t)$. We do not have the problem of leaking correlation in the Andersen's QE scheme.

The single drawback of the transformed volatility scheme is that the mean level $\theta_v = (\theta - \frac{\sigma^2}{4\kappa})v^{-1}(t)$ has a term $v^{-1}(t)$ which makes the mean level stochastic over time. The naive Euler scheme can not capture appropriately the erratic behavior of $v^{-1}(t)$ in time interval $[t, t+\Delta]$[1]. This erratic behavior of $v^{-1}(t)$ will be amplified by the negative value of θ_v for the parameter constellation $4\kappa\theta > \sigma^2$. In this case, θ_v often jumps from a large positive value to a large negative value in a short time, and vice versa. The value of $\theta_v(t_h)$ evaluated with the start value $v(t)$ in time interval $[t, t+\Delta]$ is no longer representative for the entire time interval. In order to achieve accurate simulation, we suggest the following two approximations for θ_v.

Method 1: Central Discretization

The main idea of central discretization is to displace θ_v evaluated with $v(t_h)$ by a one evaluated with $v^*(t_h)$ which is a somehow average of $v(t)$ in time interval $[t, t+\Delta]$. By usual integration, we define

$$v^*(t) = \frac{1}{\Delta}\int_t^{t+\Delta} v(s)ds \approx \frac{1}{2}[v(t) + v(t+\Delta)].$$

But $v(t+\Delta)$ is unknown at time t, it is then natural to further replace $v(t+\Delta)$ by an expected value $u(t+\Delta)$,

$$u(t+\Delta) = v(t) + \frac{1}{2}\kappa\big[(\theta - \frac{\sigma^2}{4\kappa})v^{-1}(t) - v(t)\big]\Delta.$$

Therefore we obtain the following approximated average

$$v^*(t) = \frac{1}{2}[v(t) + u(t+\Delta)]. \tag{10}$$

The new mean level is then calculated with $v^*(t)$,

$$\theta_v^*(t) = (\theta - \frac{\sigma^2}{4\kappa})/v^*(t). \tag{11}$$

The modified Euler scheme takes the following form

$$v(t_{h+1}) = v(t_h) + \frac{1}{2}\kappa\big[\theta_v^*(t_h) - v(t_h)\big]\Delta + \frac{1}{2}\sigma\sqrt{\Delta}Z_2(t_h). \tag{12}$$

This simple central discretization works much better than the naive Euler scheme (9) for the case of $4\kappa\theta \geqslant \sigma^2$. However, for the case of $4\kappa\theta < \sigma^2$, this method can not properly capture the strong erratic movements of θ_v, and produces too many biases in paths. Therefore, we recommend the central discretization only for the case of $4\kappa\theta \geqslant \sigma^2$ where the values of $v(t)$ stay most time in positive domain.

[1] By Ito's lemma, we can derive the stochastic process of $v^{-1}(t)$ which is involved in a term $v^{-3}(t)$. This indicates that the dynamics of $v^{-1}(t)$ is very bad-behaved.

Method 2: Moment-Matching

Due to the strong oscillation of θ_v between positive and negative domains, the above central discretization sometimes fails to recover the average dynamics. We propose a way to find an more robust average θ_v by moment-matching. Note $\mathbf{E}[v(t)^2] = \mathbf{E}[V(t)]$, we have

$$\mathbf{E}[v(t+\Delta)^2] = \mathbf{E}[V(t+\Delta)] = \mathbf{Var}[v(t+\Delta)] + \mathbf{E}[v(t+\Delta)]^2, \tag{13}$$

where

$$\mathbf{E}[V(t+\Delta)] = m_1(t) = \theta + (V(t) - \theta)e^{-\kappa\Delta},$$

$$\mathbf{Var}[v(t+\Delta)] = m_2(t) = \frac{\sigma_v^2}{2\kappa_v}(1 - e^{-2\kappa_v\Delta}) = \frac{\sigma^2}{4\kappa}(1 - e^{-\kappa\Delta}),$$

$$\mathbf{E}[v(t+\Delta)] = \theta_v + (v(t) - \theta_v)e^{-\kappa_v\Delta} = \theta_v + (v(t) - \theta_v)e^{-\frac{1}{2}\kappa\Delta}.$$

Rearranging the above terms yields

$$\begin{aligned} \big[\theta_v \quad &+ \quad (v(t) - \theta_v)e^{-\frac{1}{2}\kappa\Delta}\big]^2 = m_1(t) - m_2(t) \\ &= \quad \theta + (v^2(t) - \theta)e^{-\kappa\Delta} - \frac{\sigma^2}{4\kappa}(1 - e^{-\kappa\Delta}). \end{aligned} \tag{14}$$

The single unknown variable is θ_v that can be solved for as follows

$$\theta_v^*(t) = \frac{\beta - v(t)e^{-\frac{1}{2}\kappa\Delta}}{1 - e^{-\frac{1}{2}\kappa\Delta}} \tag{15}$$

with

$$\beta = \sqrt{[m_1(t) - m_2(t)]^+},$$

where β is set to zero if $m_1(t) < m_2(t)$. The matched θ_v^* can be applied for any parameter constellation and produces good simulations even for the case of $4\kappa\theta < \sigma^2$. Although this method requires a bit more computations than the central discretization method, it improves the accuracy of simulations significantly. Hence, the moment-matching method is preferred to the central discretization method, and we recommend the moment-matching method for θ_v^* in practical applications.

Some advanced simulation techniques, for example, martingale correction, can be coupled with the above scheme for potential improvements.[2]

[2]There is no Milstein scheme for $v(t)$ since the diffusion term is constant. Generally, second-order discretization method does not improve the simulation of Ornstein-Uhlenbeck process.

4 Simulation Examples

To demonstrate the quality of the proposed transformed volatility (TV) scheme, we compare the simulated prices of European call options with the analytic prices of the Heston model. Additionally, to demonstrate the performance of TV scheme in relative to other schemes, we simulate call options with three various schemes: proposed TV scheme, Andersen's QE scheme and Log-normal scheme. As shown in Andersen (2007), QE scheme outperforms other existing schemes and could be considered as benchmark method for mean-reverting square root process. Log-normal scheme is also widely used in practical applications and could be competitive to other schemes. For a systematic comparison, we consider three cases.

1. Case 1: $\kappa \geqslant \sigma^2/(2\theta)$.
2. Case 2: $\sigma^2/(2\theta) > \kappa \geqslant \sigma^2/(4\theta)$.
3. Case 3: $\kappa < \sigma^2/(4\theta)$.

The first case where $\kappa \geqslant \sigma^2/(2\theta)$ is equivalent to the condition for positive values $V(t)$. Therefore, the square root process of $V(t)$ behaves soundly, and the simulations in this case encounter normally few problems. In the second case, the parameter restriction for positive values of $V(t)$ is no longer satisfied, and this case becomes more challenging for most existing schemes. However, this case does not arise serious issues for the transformed volatility process $v(t)$ in (7) because the mean level θ_v is almost positive. The most challenging case is the third case where κ is smaller than $\sigma^2/(4\theta)$ and is far away from $\sigma^2/(2\theta)$. In this case, the most probability masses of $V(t)$ concentrate on the near of zero. This case is then a stress test case for an efficient scheme. We use the moment-matching approximation for θ_v in our TV scheme.

Table 1 gives the data in three test settings. We simulate European call prices with spot of 100, maturity of 6 years, and strikes ranging from 70 to 130. All simulations are run with a number of paths 20000. This is a moderate number from the point of theoretical view, but is compatible for practical applications. The number of time steps per year is 32 and therefore is 192 for 6 years maturity. The parameters for mean-reverting square root process are representative for equity options markets. Table 2, Table 3 and Table 4 give the numerical results using three different simulation schemes, as well as the corresponding analytical prices. For a detailed comparison, we provide also the standard deviations for the simulated prices and the relative price differences which are defined by

$$RPD = \frac{P_{Simulation} - P_{Analytic}}{P_{Analytic}}.$$

Relative price difference is preferred to absolute price difference since the former eliminates the effect of underlying spot price and strike on option

Call Option:	S_0=100	T=6Y	r=0.04	
Simulation:	Paths =20000	TS =32(per year)		
Process:	V_0=0.0225	θ=0.04	$\sigma = 0.3$	ρ=-0.5
Case 1:	$\kappa = 2$			
Case 2:	$\kappa = 0.8$			
Case 3:	$\kappa = 0.4$			

Table 1: Test data with three different κ.

price.

From these three test cases, we observe the following points:

1. The numerical results of TV and QE schemes are very close to each other through all strikes and scenarios, not only in terms of prices, but also in terms of standard deviations. This delivers an strong evidence that both schemes work very well for all parameter constellations, even for the critical test case 3.
2. The log-normal scheme is competitive to TV and GE schemes in case 1 where the square root process is good–behaved, it delivers also acceptable prices for ATM options, but produces strongly biased prices for ITM and OTM options in cases 2 and 3. The log-normal scheme fails to pass the test case 3. Both TV and QE schemes outperform log-normal scheme.
3. TV and QE produce more accurate prices for ITM options than OTM options in terms of relative price difference.
4. Both TV and QE seem to overestimate option prices slightly. No numerical results are found that both TV and QE provide lower prices than analytic option prices in the given examples.

The above numerical results and findings deliver the strong evidences that the proposed TV scheme produces highly accurate simulations for the Heston model for any parameter constellation.

5 Conclusions

In this paper we have proposed a simple approach to simulating Heston model efficiently and accurately. Instead of dealing with the original mean-reverting square root process of the variance $V(t)$, we transform the variance $V(t)$ to an equivalent volatility $v(t)$ which takes the form of a mean-reverting Ornstein-Uhlenbeck process. At the same time, we obtain also an equivalent reformulation of the Heston Model. We have shown it is simpler and more convenient to simulate the transformed Heston model than the

Strikes	TV	QE	LogN	Analytic
K=70	47.2726	47.2696	46.9707	47.1518
SDev	0.3162	0.3164	0.3388	
RPD	0.0025	0.0025	-0.0038	
K=80	40.9035	40.9014	40.6333	40.8003
SDev	0.3056	0.3059	0.3287	
RPD	0.0025	0.0025	-0.0041	
K=90	35.0888	35.0875	34.9108	34.9894
SDev	0.2926	0.2929	0.3162	
RPD	0.0028	0.0028	-0.0023	
K=100	29.8402	29.8408	29.8015	29.7543
SDev	0.2778	0.2781	0.3019	
RPD	0.0029	0.0029	0.0016	
K=110	25.1831	25.1850	25.3062	25.1049
SDev	0.2615	0.2618	0.2695	
RPD	0.0031	0.0032	0.0080	
K=120	21.1083	21.1134	21.4188	21.0302
SDev	0.2442	0.2446	0.2695	
RPD	0.0037	0.0039	0.0185	
K=130	17.5730	17.5824	18.0684	17.5020
SDev	0.2265	0.2268	0.2525	
RPD	0.0040	0.0040	0.0320	

Table 2: Test Case 1: $\kappa = 2 \geqslant \sigma^2/(2\theta)$. SDev stands for the standard deviations, RPD for the relative price difference.

Strikes	TV	QE	LogN	Analytic
K=70	47.4219	47.4210	46.8853	47.2812
SDev	0.2834	0.2835	0.3229	
RPD	0.0029	0.0030	-0.0084	
K=80	40.8673	40.8679	40.4694	40.7576
SDev	0.2730	0.2731	0.3130	
RPD	0.0026	0.0027	-0.0070	
K=90	34.7920	34.7933	34.6501	34.6872
SDev	0.2604	0.2605	0.3006	
RPD	0.0030	0.0031	-0.0011	
K=100	29.2214	29.2244	29.4391	29.1296
SDev	0.2459	0.2461	0.2861	
RPD	0.0032	0.0032	0.0106	
K=110	24.2162	24.2208	24.4391	24.1311
SDev	0.2300	0.2304	0.2703	
RPD	0.0035	0.0037	0.0298	
K=120	19.8193	19.8238	20.8727	19.7210
SDev	0.2128	0.2129	0.2533	
RPD	0.0049	0.0052	0.0584	
K=130	16.0099	16.0158	17.4480	15.9076
SDev	0.1951	0.1952	0.2359	
RPD	0.0064	0.0068	0.0968	

Table 3: Test Case 2: $\sigma^2/(4\theta) \leqslant \kappa = 0.8 < \sigma^2/(2\theta)$. SDev stands for the standard deviations, RPD for the relative price difference.

Strikes	TV	QE	LogN	Analytic
K=70	47.3616	47.3750	46.7192	47.2115
SDev	0.2559	0.2529	0.3066	
RPD	0.0032	0.0035	-0.0104	
K=80	40.6052	40.6039	40.1897	40.4726
SDev	0.2463	0.2433	0.2939	
RPD	0.0032	0.0032	-0.0070	
K=90	34.2329	34.2112	34.2369	34.0975
SDev	0.2347	0.2317	0.2816	
RPD	0.0039	0.0033	0.0041	
K=100	28.3237	28.2745	28.8903	28.1628
SDev	0.2214	0.2184	0.2672	
RPD	0.0057	0.0040	0.0258	
K=110	22.9460	22.8548	24.1725	22.7535
SDev	0.2065	0.2036	0.2512	
RPD	0.0134	0.0045	0.0623	
K=120	18.1958	18.0601	20.0733	17.9555
SDev	0.1905	0.1877	0.2341	
RPD	0.0133	0.0058	0.1179	
K=130	14.1228	13.9508	16.5605	13.8427
SDev	0.1738	0.1711	0.2164	
RPD	0.0202	0.0078	0.1963	

Table 4: Test Case 3: $\kappa = 0.4 < \sigma^2/(4\theta)$. SDev stands for the standard deviations, RPD for the relative price difference.

original one since the simulation of a mean-reverting Ornstein-Uhlenbeck process does not involve any term of square root. The only challenge is to achieve a robust approximation for the stochastic mean level. To this end, we have suggested two methods where the method based on moment-matching performs efficiently for any parameter set. Numerical examples have shown that our scheme and QE scheme produce almost identical performances not only in terms of price, but also in terms of standard deviation, and outperform other schemes such as log-normal scheme. While QE scheme suffers from the problem of "leaking correlation", transformed volatility scheme does not, and therefore, provides a high-quality alternative to the existing simulation schemes for Heston model.

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